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Hitchin functionals in $N = 2$ supergravity

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Abstract: We consider type II string theory in space-time backgrounds which admit eight supercharges. Such backgrounds are characterized by the existence of a (generically non-integrable) generalized $SU(3) \times SU(3)$ structure. We demonstrate how the corresponding ten-dimensional supergravity theories can in part be rewritten using generalised $O(6,6)$-covariant fields, in a form that strongly resembles that of four-dimensional $N = 2$ supergravity, and precisely coincides with such after an appropriate Kaluza-Klein reduction. Specifically we demonstrate that the NS sector admits a special Kähler geometry with Kähler potentials given by the Hitchin functionals. Furthermore we explicitly compute the $N = 2$ version of the superpotential from the transformation law of the gravitinos, and find its $N = 1$ counterpart.

Keywords: Extended Supersymmetry, Superstring Vacua.

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1. Introduction

The interplay between supersymmetry and geometry has been very fruitful in the past. For example, compactifications of ten-dimensional type II supergravities on Calabi-Yau threefolds Y preserve eight supercharges and yield four-dimensional $N = 2$ (ungauged) supergravities as effective low energy field theories [1−4]. The spectrum and couplings of these $N = 2$ supergravities are in turn determined by geometrical (and topological) properties of the Calabi-Yau manifolds. Supersymmetry strongly constrains the couplings and thus also constrains the Calabi-Yau geometry. For example, it implies that the moduli space of metric deformations of a Calabi-Yau manifold is the product of two special Kähler manifold characterized by two holomorphic prepotentials [1, 5 − 7]. The Calabi-Yau moduli space indeed satisfies this property and furthermore one can use geometrical methods together with mirror symmetry to compute both prepotentials exactly [8, 9].

Expanding on earlier work in refs. [10 − 12], there has recently been much interest in a generalized class of backgrounds where the Calabi-Yau manifold is replaced by a manifold Y which is no longer Ricci-flat [13]–[44]. One way such generalized compactifications arise is when localized sources (D-branes, orientifold planes) and/or background fluxes are present and the solution of the equations of motion forces the geometry to back-react to the additional background energy density. A certain class of manifolds, called 'half-flat manifolds' [45], also appeared as mirror symmetric backgrounds of type II Calabi-Yau compactifications with background fluxes [15, 18, 23].

Within this generalized set-up one is particularly interested in backgrounds which continue to preserve some of the supercharges or more generally where a number of supercurrents exist but the associated supercharges are spontaneously broken. The latter case includes examples which do not satisfy the equations of motion, such as the classical example of a Calabi-Yau manifold with generic background fluxes. In either case, the existence of the supercurrents requires that a set of spinors are globally well defined on the manifold Y which in turn implies that the structure group has to be reduced. In the mathematical literature manifolds with a reduced structure group G are called manifolds with G-structure [46, 47]. Generically G does not coincide with the holonomy group since the spinors are not necessarily covariantly constant with respect to the Levi-Civita connection. The degree to which they fail to be covariantly constant is measured by a quantity known as the intrinsic torsion and can be used to classify the G-structure.

From a particle physics point of view preserving the minimal amount of supersymmetry is the most interesting case. On a six-manifold the existence of a single globally defined spinor η requires the reduction of the structure group from Spin(6) to SU(3) and therefore manifolds with SU(3) structure play a special role. They can be characterized by the invariant spinor on the manifold or, more conveniently, by a real two-form J and a complex three-form Ω. Since η is not covariantly constant neither J nor Ω are closed. Instead $\mathrm{d}J$ and $\mathrm{d}\Omega$ decompose into $SU(3)$ representations. These define the intrinsic torsion and can be used to classify the different $SU(3)$ structures [45]. For Calabi-Yau threefolds J and Ω are closed, η is covariantly constant and the holonomy group is SU(3).

Here, we will focus on type II supergravities which have $N = 2$ supersymmetry in ten space-time dimensions. Decomposing the spinor representation in ten dimensions under $\mathrm{Spin}(1,3) \times \mathrm{Spin}(6)$ and requiring $N = 2$ supersymmetry in four dimensions implies that there are two non-vanishing spinors on Y, one for each of the original ten-dimensional spinors. Each defines an SU(3) structure. Locally, the two $SU(3)$ structures define an $SU(2)$ structure, which survives globally as an $SU(2)$ structure if the spinors never become parallel. If the spinors are always parallel we just have a single SU(3) structure.

One way to characterize this structure mathematically is in terms of "generalised geometry", first introduced by Hitchin [48]. One considers the sum of the tangent and cotangent bundle of Y, $TY \oplus T^*Y$ on which there is a natural $O(6,6)$ structure. The two six-dimensional spinors transform under a Spin(6)×Spin(6) subgroup defined by the metric and NS B-field, and being globally defined, imply that the structure group of $TY \oplus T^*Y$ actually reduces to SU(3) × SU(3) [34] (see [49] for the original, related discussion of $U(n) \times U(n)$ structures). In this formulation, the SU(3) × SU(3) structure can be defined by a sum of odd forms Φ^- and a sum of even forms Φ^+, each built out of spinor bilinears [28, 34] (see also [50] for the construction in the case of $G_2 \times G_2$ structures). From the point of view of the $TY \oplus T^*Y$ bundle these forms correspond to a pair of Spin(6, 6) spinors [51, 52, 48].

Since we are interested in backgrounds with supercurrents but, in general, spontaneously broken supersymmetry, we do not require the SU(3) structures to be integrable. Enforcing preserved supersymmetry (and the equations of motion) would impose integrability constrains [28, 34, 44]. The geometric structures used throughout this paper are therefore "almost" (or not necessarily integrable) structures. However, in order to avoid cumbersome wording, we will typically drop the "almost" when referring to them.

The description of backgrounds in terms of SU(3) structures and the generalization to SU(3) × SU(3) structures has also recently played an important role in topological string theories. In particular, it has been argued [53 – 55] that the target space theory of the A and B model topological strings can be defined in terms of a functional of the structures J or Ω first considered by Hitchin [51, 52, 48]. More generally [56] one must consider the corresponding functional for the Spin(6, 6) spinors $\Phi^\pm$. Similarly it has been possible [29, 32] to generalize the notion of topological strings away from backgrounds with SU(3)-structure (such as Calabi-Yau manifolds) to more general spaces with SU(3)×SU(3) structure, again using the spinors $\Phi^\pm$.

Returning to the physical string, for Calabi-Yau compactifications the $N = 2$ low energy effective action in four space-time dimensions can be derived by a standard Kaluza–Klein reduction where only the massless modes corresponding to harmonic forms on Y are kept [1 – 4]. This procedure is valid whenever Y is large and the supergravity approximation can be used reliably. In the presence of background fluxes the same method has been applied for example in refs. [57]-[68]. One chooses the fluxes to be small, the compactification manifold to be large and hence consistently neglects the back-reaction of the geometry. One finds that the kinetic terms are unaltered and the flux parameters appear as gauge couplings and/or mass parameters which turn the supergravity into a gauged or massive supergravity. However, when dealing with manifolds with non-integrable $SU(3)$ structure, this procedure is a bit more tricky since generically it is harder to specify in what sense one is making a small deformation. For instance, turning on H-flux on a Calabi-Yau manifold can map to a change in topology of the mirror manifold. Thus one cannot treat the intrinsic torsion easily as a simple deformation of the supergravity as was done for the fluxes.

The goal of this paper is to study type II supergravity in generic backgrounds with SU(3) (the case where the two spinors are always parallel) or, more generally, SU(3)×SU(3) structure. Our motivation is to define a 'rule' for deriving the low-energy four-dimensional effective theory and to uncover the role of the torsion in supergravity. As in Calabi-Yau compactifications this might lead to interesting insights into the interplay of geometry and supersymmetry of the effective theory. However, we begin with a more general set-up. We do not immediately confine our interest to the low energy effective action or performing a Kaluza–Klein reduction. This leads us to a reformulation of the full ten-dimensional theory, abandoning manifest ten-dimensional Lorentz invariance, but with bosonic fields transforming in $\mathrm{Spin}(1,3) \times O(6,6)$ multiplets. This is similar to and inspired by the approach pioneered in ref. [69], which considered a related reformulation of eleven-dimensional supergravity. Although we provide no direct evidence, we expect the reformulation has a local $\mathrm{Spin}(1,3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ symmetry. More specifically, we first demand that the tangent space of the ten-dimensional background is a direct sum $T^{1,3} \oplus F$ where $T^{1,3}$ is a $\mathrm{Spin}(1,3)$ bundle while F is a $\mathrm{Spin}(6)$ bundle. Then we further require that structure group of F reduces admitting an SU(3) structure. In fact we also consider the more general situation where the sum of the tangent plus the cotangent bundle admits $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. In both cases eight of the original 32 supercharges are singled out and we can rewrite the ten-dimensional supergravity with 32 supercharges in a form as if it had only eight supercharges.

In this framework, the supermultiplet structure and action follow the form of four-dimensional $N = 2$ supergravity although the theory remains fully ten-dimensional. In particular, concentrating on the bosonic fields which are scalars under $\mathrm{Spin}(1,3)$, we define a space of (not necessarily integrable) [MG: added] $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures and show, following refs. [51, 52, 48], that it admits a special Kähler geometry with a Kähler potential given by a Hitchin functional. Restricting to the particular case of a single SU(3) structure [MG: added], we furthermore rewrite the supersymmetry transformation law of the eight gravitinos in a form analogous to the transformation law of the four-dimensional $N = 2$ gravitinos. This allows us to read off the three 'Killing prepotentials' or momentum maps $\mathcal{P}^x, x = 1, 2, 3$ which are the $N = 2$ equivalent of the superpotential and the D-term. In this ten-dimensional theory they turn out to be determined by the background fluxes and the intrinsic torsion.

In the same spirit we can continue the decomposition keeping only four supercharges. In this way we find the most general $N = 1$ superpotential induced by the fluxes and torsion. We find that this generalized superpotential contains all previously known cases in appropriate limits when either torsion, NS or RR fluxes are set to zero. For example, in the torsionless case we recover the Gukov-Taylor-Vafa-Witten superpotential [70, 71, 59].

After having rewritten the ten-dimensional theory in an '$N = 2$ form' it is straightforward to perform a KK-reduction. We choose the background to be a product $M^{1,3} \times Y$ where $M^{1,3}$ is a four-dimensional manifold with Minkowskian signature while Y is a compact manifold with $SU(3)$ structure. (The more general case of compactifications with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure will be discussed elsewhere.) In the KK-reduction one conventionally keeps the light modes and integrates out the heavy ones. However, backgrounds with a generic Y do not necessarily have a flat Minkowskian ground state and the distinction between heavy and light is not straightforward. Therefore we do not specify the precise

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form of the truncation, which would depend on the particular choice of background, but instead leave it generic, extracting the set of conditions that such a reduction must satisfy to be self-consistent. The truncation is defined by extracting from the infinite tower of KK-modes only a finite subset. However, we impose one further condition in that we only keep the two gravitini in the gravitational multiplet but project out all gravitini which reside in their own (massive) spin-$\frac{3}{2}$ multiplets. This ensures that the resulting low effective action contains apart form the gravitational multiplet only $N = 2$ vector, tensor and hypermultiplets.

Once the ten-dimensional spectrum is truncated the gauge invariance of the original ten-dimensional theory is no longer automatically maintained. Instead, as we will see, gauge invariance imposes additional constraints on the truncation which also have been observed in [72]. Imposing these constraints, the $N = 2$ action takes a standard form [73, 74] – possibly with massive tensor multiplets [64, 75, 76]. This enables us to discuss in detail the supergravity/geometry correspondence. We find, as expected, that the torsion (as well as the fluxes) deform the $N = 2$ supergravity and turn it into a gauged or massive supergravity. The gauge charges and mass parameters are directly related to the fluxes and torsion and we derive the precise relationship by computing the supersymmetry transformations of the gravitino.

This paper is organized into two main sections. In section 2 we discuss the reformulation of the ten-dimensional type II supergravity theory in terms of $N = 2$–like structures while in section 3 we perform the Kaluza–Klein reduction and compute some of the couplings in the low energy effective theory. More specifically in 2.1 we first show that eight linearly realized supercharges require that the theory has a $\mathrm{Spin}(1, 3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. In section 2.2 we then show how the ten-dimensional fields decompose into $N = 2$ multiplets for the case of a single $SU(3)$ structure. After reviewing a few facts about SU(3) structures, we give the part of the action for the deformations of the NS fields in section 2.3. In section 2.4 we show that their kinetic terms form a product of two special Kähler geometries in exact analogy with the moduli space of Calabi-Yau compactifications. Furthermore the Kähler potential is determined by the sum of two Hitchin functionals both of which can be derived from a universal expression given in terms of a pure $Spin(6, 6)$ spinor [51, 48]. In section 2.5 we compute the scalar part of the supersymmetry transformations of the gravitinos and determine the ten-dimensional analog of the Killing prepotential $\mathcal{P}^x$. By an appropriate further reduction we compute the $N = 1$ superpotential in section 2.6. In section 3 we perform the KK-truncation. We first define the 'rules' for the reduction in 3.1. We project out all $\mathbf{3} \oplus \bar{\mathbf{3}}$ representations and then impose local p-form gauge invariance on the remaining spectrum. In 3.2 we discuss the reduction of the common NS-sector and show that the resulting Kähler potentials precisely coincide with the analogous Kähler potentials of Calabi-Yau manifolds. In 3.3 and 3.4 we perform the reduction of the RR-sector for type IIA and type IIB. In 3.5 we check that the 'proper' $N = 2$ Killing prepotential $\mathcal{P}^x$ obtained by truncation from its higher-dimensional 'father' agrees with the generic form dictated by $N = 2$ gauged supergravity. Finally in 3.6 we briefly discuss mirror symmetry for these generalized compactifications and 4 contains our conclusions.

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2. Type II supergravity with SU(3) × SU(3) structure

The goal in this section is to understand some of the details of how we can reformulate the ten-dimensional type II supergravity theory in terms structures analogous to $N = 2$ four-dimensional supergravity. In doing so we lose manifest Spin(1, 9) Lorentz symmetry, and instead arrange the fields in $\mathrm{Spin}(1,3) \times O(6,6)$ multiplets. We will concentrate on the scalar field part of the theory, that is multiplets which contain fields which are singlets under Spin(1, 3). In an $N = 2$ language these are the vector, tensor and hypermultiplets. In particular, we will show that there are special Kähler geometries on the spaces of scalar fields describing their kinetic terms. Furthermore, we will show how the ten-dimensional theory gauges a set of isometries on these spaces, described by a set of Killing prepotentials again just as in four-dimensional $N = 2$ supergravity. We find that all these objects can be written in a simple way in terms of generalised geometrical structures, invariant under $O(6, 6)$ transformations. In particular, the Kähler potential of the special Kähler geometry is given by the Hitchin functional.

Let us start by discussing the relation between rewriting the theory in terms of eight linearly realized supercharges ($N = 2$) and the existence of generalised SU(3) × SU(3) structures.

2.1 $N = 2$ and SU(3) × SU(3) structures

2.1.1 Effective theories and G-structures

One motivation for this paper is to consider the general low-energy gauged supergravity theory that arises when type II string theory (or rather type II supergravity) is compactified on the space-time background

$$M^{1,9} = M^{1,3} \times Y \ . \tag{2.1}$$

Here $M^{1,3}$ is the four-dimensional, physical space-time while Y is a six-dimensional compact manifold.[1] The product structure of the space-time background (2.1) implies a decomposition of the Lorentz group $\mathrm{Spin}(1,9) \supset \mathrm{Spin}(1,3) \times \mathrm{Spin}(6)$ and an associated decomposition of the spinor representation $\mathbf{16} \in \mathrm{Spin}(1,9)$ according to $\mathbf{16} \to (\mathbf{2},\mathbf{4}) \oplus (\bar{\mathbf{2}},\bar{\mathbf{4}})$.

We are interested in the situation where the effective theory on $M^{1,3}$ has the minimal $N = 2$ supersymmetry. In other words, we need to single out eight particular type II supersymmetries which descend to the effective theory. For type IIA we start with two supersymmetry parameters of opposite ten-dimensional chirality. Using a standard decomposition of the ten-dimensional gamma matrices $\Gamma^M = (\Gamma^\mu, \Gamma^m)$ as

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1} \ , \quad \mu = 0,1,2,3 \ , \qquad \Gamma^m = \gamma_5 \otimes \gamma^m \ , \quad m = 1,\dots,6 \ , \tag{2.2}$$

where $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, we can write

$$\begin{aligned} \varepsilon^1_{\mathrm{IIA}} &= \varepsilon^1_+ \otimes \eta^1_+ + \varepsilon^1_- \otimes \eta^1_- \ , \\ \varepsilon^2_{\mathrm{IIA}} &= \varepsilon^2_+ \otimes \eta^2_- + \varepsilon^2_- \otimes \eta^2_+ \ , \end{aligned} \tag{2.3}$$

[1] In this paper we do not consider the possibility of a warped background but leave the study of this class of compactification to a separate publication.

where $\varepsilon_-^{1,2} = (\varepsilon_+^{1,2})^c$ and $\eta_-^{1,2} = (\eta_+^{1,2})^c$. (Here as usual $\eta^c = D\eta^*$, where D is the intertwiner giving $-\gamma^{m*} = D^{-1}\gamma^m D$. We also have $\bar{\eta} = \eta^\dagger A$, where $\gamma^{m\dagger} = A\gamma^m A^{-1}$. By a slight abuse of notation we use plus and minus to indicate both four-dimensional and six-dimensional chiralities.) For type IIB both spinors have negative chirality resulting in the decomposition

$$\varepsilon_{\rm IIB}^A = \varepsilon_+^A \otimes \eta_-^A + \varepsilon_-^A \otimes \eta_+^A \ , \qquad A = 1, 2 \ . \tag{2.4}$$

In each case we have a pair of spinors ε_+^A in $M^{1,3}$ parameterizing the $N = 2$ supersymmetries. In addition, we have two spinors η_+^A of Y fixing precisely which of the ten-dimensional supersymmetries descend to four dimensions. Note that generically these can be different for the two ten-dimensional supersymmetry parameters ε^A.

For such a reduction to work, the spinors η_+^A must be globally defined and nowhere vanishing on Y and hence the structure group of the tangent space of Y has to reduce. Consider one such global spinor. It has to transform as a singlet under the structure group. Decomposing under $\mathrm{SU}(3) \subset \mathrm{Spin}(6)$, the complex spinor representation splits as $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$. Thus if the structure group is contained in SU(3) we indeed get a spinor singlet. Manifolds with this property are known as 'manifolds with SU(3) structure' in the mathematical literature [45]. Since we do not require the background to be supersymmetric, only that the four-dimensional effective action has a set of $N = 2$ supercurrents, there are generically no differential conditions on the spinors η_+^A. In the mathematical literature this means we have an "almost" or not necessarily integrable SU(3) structure.

From eqs. (2.3) and (2.4) we see that in general we have a pair η_+^A of such spinors, each of which defines an SU(3) structure. In summary

$$d = 4,\ N = 2 \text{ effective theory} \quad \Leftrightarrow \quad Y \text{ admits a pair of SU(3) structures} \ . \tag{2.5}$$

Locally the two spinors η_+^1 and η_+^2 span a two-dimensional subspace of the four-dimensional space of positive chirality $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ spinors. This space is invariant under $\mathrm{SU}(2) \subset \mathrm{SU}(4)$ rotations, under which both spinors are singlets. Thus locally the presence of two SU(3) structures actually implies that we have an SU(2) structure. However, globally there can be points where the spinors become parallel, and hence at these points no SU(2) structure is defined. The extreme case where the two spinors are parallel everywhere is allowed, and in this case the two SU(3) structures coincide, leading to a single SU(3) structure.

As we discuss in more detail below, a special case of a supersymmetric compactification is where Y is a Calabi-Yau manifold and $\eta_+^1 = \eta_+^2$. In this case, in deriving the low-energy effective theory, one keeps only the massless modes and disregards all heavy Kaluza–Klein modes (together with all heavy string states). However, for compactifications on generic manifolds with a pair of SU(3) structures the distinction between heavy and light modes is not straightforward. It appears that we have to define a 'rule' for the reduction to decide which modes we keep in the four-dimensional effective action and which modes we truncate away. In fact, as we now discuss, we can actually start by doing something more general, where we do not truncate the theory at all.

2.1.2 A $d = 10$ reformulation and generalized structures

The previous discussion was based on the assumption that we had a product manifold (2.1). However it is not really necessary to make such an assumption. In general, if we break the local $\mathrm{Spin}(1,9)$-invariance one can always rewrite the full $d = 10$ type II supergravity theory as though it were a theory with only eight supercharges. The structure of the theory is then analogous to $N = 2$ in four dimensions, but no Kaluza–Klein expansion is made and instead we work in ten space-time dimensions keeping all the degrees of freedom. A similar reorganization of eleven-dimensional supergravity was done in ref. [69] in order to understand the origin of hidden symmetries in lower dimensional supergravities.

More precisely, we require only that the ten-dimensional manifold $M^{1,9}$ admits a pair of SU(3) structures. This means that the tangent space $TM^{1,9}$ splits as

$$TM^{1,9} = T^{1,3} \oplus F \ , \tag{2.6}$$

where $T^{1,3}$ is a real $\mathrm{SO}(1,3)$ vector bundle and F is a SO(6) vector bundle which admits a pair of SU(3) structures. In other words we have two different decompositions of the complex vector bundle $F_{\mathbb{C}} = F \otimes \mathbb{C}$, that is

$$F_{\mathbb{C}} = E^1 \oplus \bar{E}^1 = E^2 \oplus \bar{E}^2 \ , \tag{2.7}$$

where each E^A is a complex SU(3) vector bundle corresponding to the SU(3) structure defined by η^A_+. Equivalently, recall that the original type II theory is formulated on a supermanifold $M^{1,9|16+16}$ of bosonic dimension $(1,9)$, with a manifest local $\mathrm{SO}(1,9)$ invariance and with the Grassmann variables transforming as a pair of 16-dimensional spinor representations. The requirement that we have a pair of SU(3) structures means there is a sub-supermanifold

$$N^{1,9|4+4} \subset M^{1,9|16+16} \tag{2.8}$$

still with bosonic dimension $(1,9)$, but now with only eight Grassmann variables transforming as spinors of $\mathrm{Spin}(1,3)$ and singlets of one or the other of the SU(3) groups. It is natural to reformulate the $d = 10$ supergravity in this sub-superspace. As such, although the theory is still defined in ten-dimensions (though without manifest $\mathrm{SO}(1,9)$ invariance), it will have structures analogous to those of $d = 4$, $N = 2$ supergravity, such as special Kähler moduli spaces and Killing prepotentials.

Let us now turn to a second key point. The pair of SU(3) structures are actually most naturally described as a single "generalized structure", a notion first introduced by Hitchin [48]. One starts by considering the sum of the tangent and cotangent bundles $TY \oplus T^*Y$, or more generally $F \oplus F^*$. If $v \in F$ and $\xi \in F^*$, one can see that there is a natural $O(6,6)$ metric on this space, defined by

$$(v + \xi, v + \xi) = \xi(v) \equiv \xi_m v^m \ , \tag{2.9}$$

which makes no reference to any additional structure (such as a metric) on F. Note that the metric is invariant under the diffeomorphism group $\mathrm{GL}(6,\mathbb{R}) \subset O(6,6)$ acting on F and F^*. The choice of metric g and NS two-form B can be shown to correspond to fixing an $O(6) \times$

$O(6)$ substructure. Given this substructure, the two spinors η^A_+ transform separately under the two different Spin(6) groups and the pair of SU(3) structures is actually equivalent to a (not necessarily integrable) SU(3) × SU(3) structure on $F \oplus F^*$, as discussed in ref. [34, 50]. In summary, we have argued that

$$N = 2\text{-like reformulation of type II} \quad \Leftrightarrow \quad F \oplus F^* \text{ admits a (not necessarily integrable) SU(3)} \times \text{SU(3) structure} \tag{2.10}$$

We expect that this SU(3)×SU(3) structure is actually promoted to a local symmetry of the reformulated theory, in analogy with [69]. Suppose, for instance we had compactified on a torus $Y = T^6$. It is then a familiar result that the low-energy theory has a local $O(6) \times O(6)$ symmetry and a global $O(6,6)$ symmetry, concomitant with the fact that the string theory has a T-duality symmetry [77]. For instance the scalar degrees of freedom coming from the internal metric and B-field arrange themselves into a $O(6,6)/O(6) \times O(6)$ coset. The two Spin(6) groups act separately on each spinor η^A_+. On T^6 any pair of constant spinors (η^1_+, η^2_+) parameterizes a pair of preserved supersymmetries in four dimensions and hence compactification gives an $N = 8$ effective theory. If we isolate a single pair, this can be reformulated as an $N = 2$ theory. The local $O(6) \times O(6)$ symmetry should then reduce to those symmetries that leave the pair invariant, namely a local SU(3) × SU(3) symmetry. Thus, generically we expect that the effective theory on $N^{1,9|4+4}$ will have a local Spin(1, 3) × SU(3) × SU(3) symmetry. In what follows we will however concentrate on the analog of the scalar sector of the theory and do not provide any direct evidence for this local symmetry.

In order to simplify the discussion we will frequently specialize to the case where the SU(3) × SU(3) structure defines a *global* SU(3) structure (rather than some local SU(2) structure). In other words we assume $\eta^1_+ = \eta^2_+$. This is mostly for convenience and it also allows us to connect with the existing literature on compactifications on spaces with SU(3)-structure. We stress nonetheless that from a supergravity perspective the natural framework for $N = 2$ theories and truncations is actually a generic SU(3)×SU(3) structure.

2.2 Field decompositions

Let us first look at the decomposition of the fields of type II supergravities in the sub-supermanifold $N^{1,9|4+4}$. Let us use the same notion as the previous section even though we no longer necessarily have a product manifold. A $\mu, \nu, \ldots$ index denotes the representation of a field as a tensor of $T^{1,3}$ while a $m, n, \ldots$ index denotes the representation as a tensor of F. In addition, we specialize to the case of a global SU(3) structure. This means we can futher decompose the F-tensors into into SU(3) representations.

The common NS sector contains the metric g_{MN}, an antisymmetric tensor B_{MN} and the dilaton ϕ. Their decomposition into SU(3) representation is displayed in table 1. The notation $\mathbf{a_b}$ denotes a field in the SU(3) representation $\mathbf{a}$ and with four-dimensional spin $\mathbf{b}$, with $\mathbf{T}$ denoting an antisymmetric tensor or pseudo-scalar. The representations are real except for $\mathbf{6}$ and $\mathbf{3}$ and their conjugates.

The RR sector of type IIA supergravity features a vector C_M and a three-form C_{MNP}; their decompositions are given in table 2.

g_{MN}	$g_{\mu\nu}$	$\mathbf{1_2}$
	$g_{\mu m}$	$(\mathbf{3}+\bar{\mathbf{3}})_\mathbf{1}$
	g_{mn}	$\mathbf{1_0} + (\mathbf{6}+\bar{\mathbf{6}})_\mathbf{0} + \mathbf{8_0}$
B_{MN}	$B_{\mu\nu}$	$\mathbf{1_T}$
	$B_{\mu m}$	$(\mathbf{3}+\bar{\mathbf{3}})_\mathbf{1}$
	B_{mn}	$\mathbf{1_0} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{0} + \mathbf{8_0}$
ϕ		$\mathbf{1_0}$

Table 1: Decomposition of the NS sector in $SU(3)$ representations.

C_M	C_μ	$\mathbf{1_1}$
	C_m	$(\mathbf{3}+\bar{\mathbf{3}})_\mathbf{0}$
C_{MNP}	$C_{\mu\nu p}$	$(\mathbf{3}+\bar{\mathbf{3}})_\mathbf{T}$
	$C_{\mu np}$	$\mathbf{1_1} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{1} + \mathbf{8_1}$
	C_{mnp}	$(\mathbf{1}+\mathbf{1})_\mathbf{0} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{0} + (\mathbf{6}+\bar{\mathbf{6}})_\mathbf{0}$

Table 2: Type IIA decomposition of the RR sector in $SU(3)$ representations.

$C^{(0)}$		$\mathbf{1_0}$
C_{MN}	$C_{\mu\nu}$	$\mathbf{1_T}$
	$C_{\mu m}$	$(\mathbf{3}+\bar{\mathbf{3}})_\mathbf{1}$
	C_{mn}	$\mathbf{1_0} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{0} + \mathbf{8_0}$
C_{MNPQ}	$C_{\mu npq}$	$\frac{1}{2}\left[(\mathbf{1}+\mathbf{1})_\mathbf{1} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{1} + (\mathbf{6}+\bar{\mathbf{6}})_\mathbf{1}\right]$
	$C_{mnpq}/C_{\mu\nu mn}$	$\mathbf{1_0} + (\mathbf{3}+\bar{\mathbf{3}})_\mathbf{0} + \mathbf{8_0}$

Table 3: Type IIB decomposition of the RR sector in $SU(3)$ representations.

In type IIB one has a scalar C_0, a two-form C_2 and a four-form C_4 with a self-dual field strength F_5. Their decompositions are recorded in table 3. The self-duality of F_5 relates $C_{\mu\nu mn}$ to C_{mnpq}, and leaves only half of the representations in $C_{\mu npq}$ as independent fields.

Finally let us turn to the fermionic sector which contains two gravitinos Ψ_M and two dilatinos λ. In type IIA they have opposite ten-dimensional chirality while in IIB they have the same chirality. The 16-dimensional spinor representation decomposes according

Ψ_M	Ψ_μ	$\mathbf{1_{3/2}} + \mathbf{3_{3/2}}$
	Ψ_m	$\mathbf{1_{1/2}} + \mathbf{3_{1/2}} + 2\,\bar{\mathbf{3}}_{\mathbf{1/2}} + \mathbf{6_{1/2}} + \mathbf{8_{1/2}}$
λ		$\mathbf{1_{1/2}} + \mathbf{3_{1/2}}$

Table 4: Decomposition of the fermions in $SU(3)$ representations.

to

$$\mathbf{16} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3}) \oplus (\bar{\mathbf{2}}, \mathbf{1}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{3}}) \; . \tag{2.11}$$

This in turn leads to the decompositions displayed in table 4. (Here all the representations are assumed to be complex.)

Altogether these fields can be assembled into $N = 2$ multiplets. In both theories one finds a gravitational multiplet, six spin-$\frac{3}{2}$ multiplets, 15 vector multiplets, nine hypermultiplets and one tensor multiplet. (Of course, altogether these $N = 2$ multiplets precisely fit into a single $N = 8$ gravitational multiplet.) The distribution of the fields into $N = 2$ multiplets is not uniquely determined by their $SU(3)$ representation. However, we are mostly interested in the case where only the two gravitinos in the gravitational multiplet are kept while the six extra spin-$\frac{3}{2}$ multiplets are projected out or become massive.[2] In this case one has a 'standard' $N = 2$ theory in that only vector, tensor and hypermultiplets coupled to the gravitational multiplet with known couplings occur. This situation can be arranged if one removes all triplets from the spectrum. In this case the distribution of the fields into $N = 2$ multiplets is uniquely determined by their SU(3) quantum numbers.

For type IIA we display the multiplets in table 5 while the type IIB multiplets are given in table 6. We see that the $SU(3)$ representations are permuted between type IIA and type IIB which expresses the fact vector- and hypermultiplets are exchanged under mirror symmetry.[3]

2.3 Scalar action: kinetic terms

From now on, we will concentrate on the scalar part of the generalized action. As we have seen in the previous section, the relevant NS components are ϕ, $B_{\mu\nu}$, g_{mn} and B_{mn}. Note that g_{mn} enters both the vector multiplets and hypermultiplets. To distinguish the two contributions we first need to understand the geometry on the space of metrics on manifolds with SU(3) $\times$ SU(3) structure. As before we will actually specialize to the SU(3) case. Thus as a prerequisite it is useful to recall some facts about SU(3) structures. In what follows we will consider the analog of the four-dimensional kinetic terms for these scalar degrees of freedom. The analog of the potential term will be discussed in section 2.5.

[2] Recall that a massless $N = 2$ spin-$\frac{3}{2}$ multiplet contains a spin-$\frac{3}{2}$ gravitino, two vectors and a Weyl fermion, while a massive spin-$\frac{3}{2}$ multiplet contains a spin-$\frac{3}{2}$ gravitino, four vectors, six Weyl fermions and four scalars.

[3] Strictly speaking in type IIB one has the choice to assemble the spectrum either in tensor or hypermultiplets. If they are massless one can always dualize the tensor to a hypermultiplet. However for massive multiplets such a procedure is not straightforward and it often more convenient to keep the tensors in the spectrum [64, 75].

multiplet	$SU(3)$ rep.	field content
gravity multiplet	$\mathbf{1}$	$(g_{\mu\nu}, C_\mu, \Psi_\mu)$
tensor multiplet	$\mathbf{1}$	$(B_{\mu\nu}, \phi, C_{mnp}, \lambda)$
vector multiplets	$\mathbf{8}+\mathbf{1}$	$(C_{\mu np}, g_{mn}, B_{mn}, \Psi_m)$
hypermultiplets	$\mathbf{6}$	$(g_{mn}, C_{mnp}, \Psi_m)$

Table 5: $N = 2$ multiplets in type IIA.

multiplet	$SU(3)$ rep.	field content
gravity multiplet	$\mathbf{1}$	$(g_{\mu\nu}, C_{\mu npq}, \Psi_\mu)$
double tensor multiplet	$\mathbf{1}$	$(B_{\mu\nu}, C_{\mu\nu}, \phi, C^{(0)}, \lambda)$
vector multiplets	$\mathbf{6}$	$(C_{\mu npq}, g_{mn}, \Psi_m)$
hypermultiplets	$\mathbf{8}+\mathbf{1}$	$(g_{mn}, B_{mn}, C_{mn}, C_{mnpq}, \Psi_m)$

Table 6: $N = 2$ multiplets in type IIB.

2.3.1 Geometry of SU(3) structures

One way to define manifolds with G-structure is to demand the existence of a G-invariant tensor or spinor on the manifold [46, 47]. We have argued that an invariant spinor η_+ corresponds to picking out a particular supersymmetry in the type II theory. Given η_+ defining an SU(3)-structure we can also build a set of SU(3)-invariant forms. These are constructed as follows. Using the six-dimensional gamma-matrices γ^m defined in (2.2) we can construct a globally defined two-form J and a complex three-form Ω_η given by

$$\bar\eta_\pm \gamma^{mn}\eta_\pm = \pm\frac{1}{2}\mathrm{i}\, J^{mn}\ , \qquad \bar\eta_-\gamma^{mnp}\eta_+ = \frac{1}{2}\mathrm{i}\,\Omega_\eta^{mnp}\ , \qquad \bar\eta_+\gamma^{mnp}\eta_- = \frac{1}{2}\mathrm{i}\,\bar\Omega_\eta^{mnp}\ . \tag{2.12}$$

Here $\eta_\pm$ are normalized so that $\bar\eta_\pm\eta_\pm = \frac{1}{2}$ and $\gamma^{m_1\ldots m_p} = \gamma^{[m_1}\gamma^{m_2}\ldots\gamma^{m_p]}$ are anti-symmetrized products of six-dimensional γ-matrices.[4] Using appropriate Fierz identities one shows that with this normalization for the spinors, J and Ω_η are not independent but satisfy

$$J\wedge J\wedge J = \frac{3}{4}\mathrm{i}\,\Omega_\eta\wedge\bar\Omega_\eta\ , \qquad J\wedge\Omega_\eta = 0\ . \tag{2.13}$$

Furthermore, raising an index on J defines an almost complex structure I in that it satisfies $I^2 = -\mathbf{1}$. With respect to this almost complex structure J is a $(1,1)$-form while Ω_η is a $(3,0)$-form.

[4] By Ω_η we denote the three-form defined in (2.12) which is built from normalized spinors η. Later on in this paper we will also introduce the three-form Ω which obeys a different normalization.

It is helpful in what follows to note that one can actually *define* the SU(3) structure in terms of a pair of *real* forms (J, ρ) where $\rho = \mathrm{Re}\,\Omega_\eta$. The forms cannot be arbitrary but must be *stable* [52]. This means that they live in an open orbit under the action of general transformations $\mathrm{GL}(6,\mathbb{R})$ in the tangent space at each point. A stable two-form J then defines a $\mathrm{Sp}(6,\mathbb{R})$ structure while a stable real form ρ defines a $\mathrm{SL}(3,\mathbb{C})$ structure. Together they define an SU(3) structure provided the embeddings of $\mathrm{Sp}(6,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{C})$ in $\mathrm{GL}(6,\mathbb{R})$ are compatible. This requires

$$J \wedge J \wedge J = \frac{3}{2}\rho \wedge \hat{\rho}\ , \qquad J \wedge \rho = 0\ , \tag{2.14}$$

where $\hat{\rho} = \mathrm{Im}\,\Omega_\eta = *\rho$. As we will discuss in section 2.4.5 $\hat{\rho}$ is actually determined by ρ, independent of J. Note that since $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ the pair (J, ρ) satisfying (2.14) also defines an SO(6) structure and hence a metric.

Returning to the spinor η_+, if the corresponding supercharge is conserved in a space-time background where all fields vanish other than the metric, then the Killing spinor equations imply that η_+ is covariantly constant with respect to the Levi-Civita connection. Geometrically this says that the holonomy of $M^{1,9}$ is in SU(3). This means we have a metric product of the form (2.1) where $M^{1,3}$ is flat $\mathbb{R}^{1,3}$ and Y is a Calabi-Yau manifold.

However, if one also allows for the possibility of spontaneously broken supercharges, while we still have an SU(3) structure, η_+ is no longer required to be covariantly constant. Nonetheless, for any η_+, one can always find a torsionful connection $\nabla^{(T)}$ on the six-dimensional vector bundle F, which is compatible with the metric to g_{mn} and which obeys $\nabla^{(T)}\eta = 0$. Calabi-Yau manifolds are thus special cases of manifolds with $SU(3)$ structure for which the torsion vanishes. The part of the torsion which is independent of the choice of $\nabla^{(T)}$ is known as the "intrinsic torsion" and can be used to classify the types of SU(3) structure. Since η is no longer covariantly constant both J and Ω_η are no longer closed but instead they obey [45]

$$\begin{aligned} \mathrm{d}J &= \frac{3}{4}\mathrm{i}\left(W_1\bar{\Omega}_\eta - \bar{W}_1\Omega_\eta\right) + W_4 \wedge J + W_3\ , \\ \mathrm{d}\Omega_\eta &= W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega_\eta\ , \end{aligned} \tag{2.15}$$

with

$$W_3 \wedge J \;=\; W_3 \wedge \Omega_\eta \;=\; W_2 \wedge J^2 \;=\; 0\ . \tag{2.16}$$

One finds that the five different W's completely determine the intrinsic torsion. Note that W_1 is a zero-form, W_4, W_5 are one-forms, W_2 is a two-form and W_3 is a three-form and each can be characterized by its SU(3) transformation properties. Calabi-Yau manifolds are manifolds of SU(3) structure where all five torsion classes vanish. Any subset of vanishing torsion classes on the other hand define specific classes of SU(3) structure manifolds.

Now we turn to the effective action for the metric degrees of freedom or, as we will see, the effective action for J and ρ.

2.3.2 Kinetic terms for NS deformations

In order to separate the vector multiplet and hypermultiplet degrees of freedom it is better to work in terms of the SU(3)-structure (J, ρ) rather than the metric g_{mn}. Decomposing deformations of the structure into SU(3)-representations, given the constraints (2.14), one can write

$$\begin{aligned} \delta J &= \lambda J + i_v \rho + K \ , \\ \delta\rho &= \frac{3}{2}\lambda\rho + \gamma\hat{\rho} - v \wedge J + M \ . \end{aligned} \tag{2.17}$$

where λ and γ are scalars, v is a real vector, transforming as $\mathbf{3} + \bar{\mathbf{3}}$, K a is real two-form transforming as $\mathbf{8}$ and M is a (primitive) three-form transforming as $\mathbf{6} + \bar{\mathbf{6}}$. This implies that

$$\begin{aligned} \rho \wedge K \ &= \ J \wedge J \wedge K \ = \ 0 \ , \\ J \wedge M \ &= \ \rho \wedge M \ = \ \hat{\rho} \wedge M \ = \ 0 \ . \end{aligned} \tag{2.18}$$

Recalling that the SU(3) structure defines the metric g_{mn}, it is easy to show that the two sets of deformations are related by

$$\delta g_{mn} = \lambda g_{mn} - J_m{}^p K_{pn} - \frac{1}{2}\rho_m{}^{pq} M_{pqn} \ . \tag{2.19}$$

Here we see explicitly that δg_{mn} contains scalar, $\mathbf{8}$ and $\mathbf{6} + \bar{\mathbf{6}}$ deformations as we already noted in table 1. (By definition $J_m{}^p K_{pn}$ and $\rho_m{}^{pq} M_{npq}$ are symmetric on m and n.)

Clearly, there are more degrees of freedom in the SU(3)-structure than in the metric. This is not surprising since the metric parameterizes the coset $\mathrm{GL}(6,\mathbb{R})/\mathrm{SO}(6)$ which is 21-dimensional, while the pair (J,ρ) parameterizes $\mathrm{GL}(6,\mathbb{R})/\mathrm{SU}(3)$ which is 28-dimensional. The vector v and scalar γ represent the extra seven parameters: deformations which change the SU(3)-structure but leave the metric invariant. It is natural, since we have local SU(3) symmetry, to formulate the theory in terms of J and ρ. However, we expect to find a local symmetry removing the non-metric degrees of freedom represented by v and γ. This is the remnant of the local $\mathrm{Spin}(6) \subset \mathrm{Spin}(9,1)$ rotational symmetry of the vierbein formulation of gravity.

Note, in addition, that the vector deformation v transforms as $\mathbf{3} + \bar{\mathbf{3}}$ as do the additional spin-$\frac{3}{2}$ degrees of freedom coming form Ψ_μ discussed in section 2.2. If we want to consistently restrict to a 'standard' $N = 2$ theory and ignore these additional spin-$\frac{3}{2}$ fields we must drop all such triplet representations and hence set v to zero. We will often do this in what follows and only in section 2.4.6 discuss some properties of the more general case.

Now let us finally turn to the kinetic terms for g_{mn}, B_{mn} and ϕ. The point is that, given the split (2.6), we can always decompose the derivative operator ∂_M into a part on $T^{1,3}$ and a part on F labeled ∂_μ and ∂_m respectively,

$$\partial_M = (\partial_\mu, \partial_m). \tag{2.20}$$

Any term in the ten-dimensional theory with two ∂_μ operators we denote as kinetic, while any scalar field term with no such operators we denote as contributing to the potential.

As when conpactifying, we need to rescale the four-dimensional part of the metric $g_{\mu\nu}$ and also define a new "four-dimensional" dilaton in order to diagonalize the kinetic terms and get a conventional Einstein term. In analogy with the case of compactification on a Calabi-Yau manifolds we write

$$g^{(4)}_{\mu\nu} = \mathrm{e}^{-2\phi^{(4)}} g_{\mu\nu} \ , \qquad \phi^{(4)} = \phi - \frac{1}{4}\ln\det g_{mn} \ , \tag{2.21}$$

where g_{mn} is not rescaled but taken in the ten-dimensional string frame. Note that these definitions imply that

$$\begin{aligned} e^{-2\phi^{(4)}} &\in \det F^* = \Lambda^6 F^* \ , \\ g^{(4)}_{\mu\nu} &\in T^{1,3\,*} \otimes T^{1,3\,*} \otimes \Lambda^6 F^* \ . \end{aligned} \tag{2.22}$$

This means for instance that the exponential of the four-dimensional dilaton transforms as a six-form under $GL(6,\mathbb{R})$ transformations on F.

The bosonic NS part of the ten-dimensional type II action reads

$$S_{NS} = \int d^{10}x \sqrt{g}\, e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] \ . \tag{2.23}$$

Keeping only terms with ∂_μ and also only Spin$(1,3)$-scalar fields we find, given the redefinitions (2.21),

$$\begin{aligned} S_{NS} = \int d^{10}x \sqrt{g^{(4)}} \big[& R^{(4)} - 2(\partial\phi^{(4)})^2 - \frac{1}{12} e^{-4\phi^{(4)}} H^2_{(4)} \\ & - \frac{1}{4} g^{mp} g^{nq} (\partial_\mu g_{mn} \partial^\mu g_{pq} + \partial_\mu B_{mn} \partial^\mu B_{pq}) + \dots \big] \ , \end{aligned} \tag{2.24}$$

where $H^{(4)}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$ and all contractions are with $g^{(4)}_{\mu\nu}$. Note that, for instance, $\sqrt{g^{(4)}} R^{(4)} \in \det T^{1,3\,*} \otimes \det F^*$ and hence (2.24) does transform properly as a ten-dimensional generally covariant expression. The first three terms give the usual kinetic terms of the four-dimensional metric $g^{(4)}$ and the $B_{\mu\nu}$–ϕ part of the tensor or double tensor multiplet. The last two terms in (2.24) define a metric of the space of metric and B-field deformations [7]. We write, given the expansions (2.17)

$$\begin{aligned} ds^2 &= \frac{1}{8} g^{mp} g^{nq} (\delta g_{mn} \delta g_{pq} + \delta B_{mn} \delta B_{pq}) \\ &= \left[\frac{3}{4} \delta\lambda\delta\lambda + \frac{1}{8} g^{mp} g^{nq} \delta K_{mn} \delta K_{pq} + \frac{1}{8} g^{mp} g^{nq} \delta B_{mn} \delta B_{pq} \right] \\ &\qquad + \frac{1}{24} g^{mr} g^{ns} g^{pt} \delta M_{mnp} \delta M_{rst} \\ &\equiv ds^2(J,B) + ds^2(\rho). \end{aligned} \tag{2.25}$$

Note that without any truncation this metric is precisely the metric on the Narain coset $O(6,6)/O(6) \times O(6)$ [77]. The vector δv_m and scalar $\delta\gamma$ deformations of J and ρ do not enter (2.25) because they represent non-metric degrees of freedom, that is deformations which change the SU(3) structure but leave the metric unchanged.

The derivation of (2.24) and (2.25) is completely analogous to the derivation given for example in refs. [3, 7] for Calabi-Yau compactifications. The difference here is that we do not assume any compactification and keep the dependence on all ten space-time coordinates. Furthermore, the background is not a Calabi-Yau but only constrained to have SU(3) or SU(3) $\times$ SU(3) structure.

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The exact same decomposition of the ten-dimensional action can be done for the RR-part of the respective actions for type IIA and type IIB. Again this is in complete analogy with the derivation of the four-dimensional action for Calabi-Yau compactifications performed in [3, 4]. This results in the kinetic terms for the gauge bosons and the kinetic terms for the RR scalars and tensors. We will not do this explicitly here since the couplings to the SU(3) structure J and ρ is exactly the same as for Calabi-Yau compactifications and thus we can borrow the results from the literature [3, 4].

Instead in the following sections we will show that $\mathrm{d}s^2(J, B)$ and $\mathrm{d}s^2(\rho)$ correspond to special Kähler metrics on the moduli space of $B + \mathrm{i}J$ and ρ respectively, and, in addition, how these structures are intimately related to Spin(6, 6) spinors and Hitchin functionals.

2.4 Special Kähler manifolds and stable forms

In this section we review, essentially following Hitchin [52], how the spaces of forms J and ρ defining the SU(3) structure each separately admit a natural special Kähler metric. We then discuss the corresponding structure on the space of SU(3) metrics g_{mn} (that is metrics which are compatible with some SU(3) structure). This is done by first noting that both $\mathrm{e}^{\mathrm{i}J}$ and $\rho + \mathrm{i}\hat{\rho}$ are pure Spin(6, 6) spinors and using Hitchin's result that there is a natural special Kähler metric on the space of (stable) real Spin(6, 6) spinors. The special Kähler metrics on the spaces of J and ρ then arise as special cases. In each case they agree with the metrics (2.25) we found directly from rewriting the type II supergravity theory.

Concretely this section is organized as follows. We first briefly review the notion of special Kähler geometry in 2.4.1. Then in section 2.4.2 we discuss in more detail the properties of SU(3)×SU(3) structures and in particular show that they can be conveniently expressed in terms of pure Spin(6, 6) spinors. In 2.4.3 we review (in a slightly different language) Hitchin's result that there is a natural special Kähler metric on the space of (stable) real Spin(6, 6) spinors. The special Kähler metrics for ρ and J will then arise as special cases in sections 2.4.4 and 2.4.5. Finally in 2.4.6 we discuss the geometry of the corresponding constrained space of SU(3) metrics.

2.4.1 Review of special Kähler geometry

First let us briefly recall the structure of special Kähler geometry [73, 5, 78, 79, 74]. There are two types of special Kähler structures which we will now summarize.

In globally supersymmetric $N = 2$ theories the scalar fields in the vector multiplets can be viewed as coordinates of a *rigid* special Kähler manifold. This implies one has $(2n)$-dimensional Kähler manifold U with a *flat* holomorphic $\mathrm{Sp}(2n, \mathbb{R})$ bundle $\mathcal{G}$ with a holomorphic section Φ such that the Kähler potential is given by

$$K_{\text{rigid}} = \mathrm{i}\omega\big(\Phi, \bar{\Phi}\big) , \qquad \omega(\partial\Phi, \partial\Phi) = 0 . \tag{2.26}$$

where $\omega(\cdot, \cdot)$ is the symplectic product on $\mathcal{G}$ and ∂ is the holomorphic derivative on U. One can typically introduce holomorphic coordinates Z^I on U and holomorphic functions $F_I(Z)$ such that

$$\omega\big(\Phi, \bar{\Phi}\big) = \bar{Z}^I F_I - Z^I \bar{F}_I \ , \tag{2.27}$$

with $I = 1, \dots, n$. The second condition in (2.26) implies that locally we can write $F_I = \partial F / \partial Z^I$ where the holomorphic function $F(Z)$ is known as the *prepotential*.

In $N = 2$ supergravity the same scalar fields in the vector multiplets are coordinates of a *local* special Kähler manifold. The latter is a Hodge-Kähler manifold $\mathcal{M}$ of real dimension $2n$ together with a line bundle $\mathcal{L}$ and a holomorphic $\mathrm{Sp}(2n+2, \mathbb{R})$ vector bundle $\mathcal{H}$ over $\mathcal{M}$. One requires that $\mathcal{L}$ embeds holomorphically in $\mathcal{H}$. In addition there exists a holomorphic section Φ of $\mathcal{L}$ such that the Kähler potential is given by

$$K \ = \ -\ln \mathrm{i}\omega\big(\Phi, \bar{\Phi}\big) \ , \qquad \omega(\Phi, \partial\Phi) \ = \ 0 \ , \tag{2.28}$$

where now $\omega(\cdot, \cdot)$ is the symplectic product on $\mathcal{H}$ and ∂ is the holomorphic derivative on $\mathcal{M}$. The section Φ is not unique but can be shifted by a holomorphic gauge transformation on $\mathcal{L}$ corresponding to a Kähler transformation. Just as in the rigid case one can introduce holomorphic coordinates Z^I on $\mathcal{H}$ together with a set of holomorphic functions $F_I(Z)$ such that (2.27) holds, though now with $I = 0, 1, \dots, n$. Again, the second condition in (2.28) implies $F_I = \partial F / \partial Z^I$. However, the prepotential $F(Z)$ is now homogeneous of degree two. Locally one can introduce special holomorphic coordinates $z^i = Z^i / Z^0, i = 1, \dots, n$ on $\mathcal{M}$. In these coordinates the prepotential has the form

$$F(Z) = (Z^0)^2 \, f(z^i) \ , \tag{2.29}$$

where $f(z^i)$ is a function of the z^i.

2.4.2 $O(6,6)$ spinors and $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures

We argued in section 2.1 that in general $N = 2$ supersymmetry (or eight linearly realized supercharges) leads to a theory with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure, by which we mean a pair of $\mathrm{SU}(3)$ structures. From one perspective this simply says that we have a pair of spinors η_+^A, $A = 1, 2$, or equivalently a pair of structures (J^A, ρ^A) which are given in terms of the η_+^A exactly as in section 2.3.1. However, it turns out that it is convenient to reformulate these structures in terms of $\mathrm{Spin}(6,6)$ spinors [49, 48] since it makes the local $\mathrm{SU}(3) \times \mathrm{SU}(3)$ symmetry of the theory manifest [34, 50]. In this section we briefly review this reformulation as it will be essential for showing the special Kähler structure of the metric (2.25).

Let $X = v + \xi$ be an element of $F \oplus F^*$ where $v \in F$ and $\xi \in F^*$. Recall that there is a natural $O(6,6)$ metric on $F \oplus F^*$ given by (2.9). The basic $\mathrm{Spin}(6,6)$ spinor representations are Majorana–Weyl. In fact, the two chiral spinor bundles S^+ and S^- are isomorphic to the space of even and odd forms respectively

$$S^+ \simeq \Lambda^{\mathrm{even}} F^*, \qquad S^- \simeq \Lambda^{\mathrm{odd}} F^*. \tag{2.30}$$

Under this isomorphism, the Clifford action is realized by

$$(v + \xi) \cdot \chi^\pm = i_v \chi^\pm + \xi \wedge \chi^\pm \tag{2.31}$$

for $\chi^\pm \in \Lambda^{\mathrm{even/odd}} F^*$ and where i_v denotes contraction with the vector v. More explicitly, let f_m with $m = 1, \dots, 6$ be a basis on F and let e^m be the dual basis on F^* (that is

$e^m(f_n) = \delta^m{}_n$). We can write the $O(6,6)$ gamma matrices Γ^Σ where $\Sigma = 1, \dots, 12$ as

$$X_\Sigma \Gamma^\Sigma = v^m \Gamma_m + \xi_m \Gamma^m \quad \text{with} \quad \Gamma_m = i_m \ , \quad \Gamma^m = e^m \wedge \ , \tag{2.32}$$

where we use the shorthand i_m for i_{f_m}.

The isomorphism between spinors and forms is not canonical. If one considers carefully how the spinors transform under the $\mathrm{GL}(6,\mathbb{R})$ subgroup one finds[5]

$$S^\pm \simeq \Lambda^{\text{even/odd}} F^* \otimes |\det F|^{1/2} \ . \tag{2.33}$$

To specify the isomorphism one needs to choose an element of $|\det F|^{1/2}$. Since one takes the absolute value this bundle is trivial. If the manifold is orientable (as will be the case for us) the isomorphism can equivalently be fixed by choosing a particular volume form $\epsilon \in \Lambda^6 F^* = \det F^*$. Specifically we define the isomorphism

$$\begin{aligned} f_\epsilon : S^\pm &\to \Lambda^{\text{even/odd}} F^*, \\ u^\pm &\mapsto \chi^\pm = u^\pm \sqrt{\epsilon}. \end{aligned} \tag{2.34}$$

In what follows we will actually be interested in forms $\chi^\pm \in \Lambda^{\text{even/odd}} F^*$. We will use

$$\chi_\epsilon = f_\epsilon^{-1}(\chi^\pm) \tag{2.35}$$

to denote the corresponding element in $S^\pm$. We will also often refer to the forms $\chi^\pm$ as "spinors" assuming that the isomorphism (2.34) is understood.

As usual, the intertwiner C between Γ^Σ and its transpose representation $-(\Gamma^\Sigma)^{\mathrm{T}} = C^{-1}\Gamma^\Sigma C$ defines a bilinear form on $S^\pm$. Explicitly if α, β are $\mathrm{Spin}(6,6)$ spinor indices and $u, v \in S^\pm$ then the inner product is given by

$$\bar{u}^\pm v^\pm \equiv (C^{-1})_{\alpha\beta}\, u^{\pm\,\alpha} v^{\pm\,\beta} \ . \tag{2.36}$$

The inner prduct between spinors in S^+ and S^- vanishes because the spinors have different chirality. Since here $C^{\mathrm{T}} = -C$ the bilinear form actually defines a symplectic structure on $S^\pm$. Thus we will also often write it as

$$\omega\big(u^\pm, v^\pm\big) \equiv \bar{u}^\pm v^\pm, \tag{2.37}$$

so that in components $\omega_{\alpha\beta} = C^{-1}_{\alpha\beta}$.

Given the isomorphism between $S^\pm$ and $\Lambda^{\text{even/odd}}$ there is a corresponding inner product $\langle \psi^\pm, \chi^\pm \rangle$ on the space of forms known as the Mukai pairing. If the isomorphism is defined by the volume form ϵ one defines

$$\begin{aligned} \big\langle \psi^+, \chi^+ \big\rangle &\equiv \omega\big(\psi^+_\epsilon, \chi^+_\epsilon\big)\, \epsilon = \psi^+_0 \wedge \chi^+_6 - \psi^+_2 \wedge \chi^+_4 + \psi^+_4 \wedge \chi^+_2 - \psi^+_6 \wedge \chi^+_0 \ , \\ \big\langle \psi^-, \chi^- \big\rangle &\equiv \omega\big(\psi^-_\epsilon, \chi^-_\epsilon\big)\, \epsilon = -\psi^-_1 \wedge \chi^-_5 + \psi^-_3 \wedge \chi^-_3 - \psi^-_5 \wedge \chi^-_1 \ , \end{aligned} \tag{2.38}$$

[5]Note that there is actually always a second spin structure, twisted not by $|\det F|^{1/2}$ but by $\det F |\det F|^{-1/2}$. These differ by a sign in the action of elements of $\mathrm{GL}(6,\mathbb{R})$ with negative determinant.

where the subscripts denote the degree of the component forms in $\Lambda^{\text{even/odd}}F^*$. Note that the Mukai pairing is independent of the particular choice of volume form ϵ. However it is also not quite a symplectic structure on $\Lambda^{\text{even/odd}}F^*$ since it is a map to Λ^6F^* and not to $\mathbb{R}$.

From the metric g_{mn} one identifies the subgroup of diffeomorphisms which preserves g_{mn} as $O(6) \subset \mathrm{GL}(6,\mathbb{R}) \subset \mathrm{Spin}(6,6)$. Given a spin structure, we can then decompose $u^+ \in S^+$ into representations of the corresponding Spin(6) group according to

$$u^+ = \zeta_+ \otimes \bar{\zeta}'_+ + \zeta_- \otimes \bar{\zeta}'_- \ , \tag{2.39}$$

where ζ_+ and ζ'_+ are positive chirality (complex) spinors of Spin(6). For $u^- \in S^-$ one finds

$$u^- = \zeta_+ \otimes \bar{\zeta}'_- + \zeta_- \otimes \bar{\zeta}'_+ \ . \tag{2.40}$$

In both cases the Spin(6) acts on both the left and the right.

The volume form ϵ_g defined by the metric and spin structure, provides a natural isomorphism with $\Lambda^{\text{even/odd}}F^*$. This can be seen directly by taking Fierz identities. In particular, if γ^m are the gamma matrices for the Spin(6), one has

$$\zeta_+ \otimes \bar{\zeta}'_\pm = \frac{1}{4}\sum_{k=0}^{6}\frac{1}{k!}\left(\bar{\zeta}'_\pm \gamma_{m_1\dots m_k}\zeta_+\right)\gamma^{m_k\dots m_1}, \tag{2.41}$$

showing that any given Spin(6,6) spinor is equivalent to a set of k-forms $\bar{\zeta}'_\pm \gamma_{m_1\dots m_k}\zeta_+$. Explicitly we have the isomorphism

$$\zeta_+ \otimes \bar{\zeta}'_\pm \sqrt{\epsilon_g} = \frac{1}{4}\bigoplus_{k=0}^{6}\frac{1}{k!}\left(\bar{\zeta}'_\pm \gamma_{m_1\dots m_k}\zeta_+\right) e^{m_k}\wedge\dots\wedge e^{m_1}. \tag{2.42}$$

Let us now rewrite a given SU(3) structure in this formalism. Recall the SU(3) structure was defined by a spinor η_+ with $\eta_- = \eta^c_+$. This allows us to define two complex Spin(6,6) spinors as [28]

$$\begin{aligned} \eta_+ \otimes \bar{\eta}_+ \sqrt{\epsilon_g} &= \frac{1}{8}\mathrm{e}^{-\mathrm{i}J} \in \Lambda^{\text{even}}F^*_{\mathbb{C}} \ , \\ \eta_+ \otimes \bar{\eta}_- \sqrt{\epsilon_g} &= -\frac{1}{8}\mathrm{i}\Omega_\eta \in \Lambda^{\text{odd}}F^*_{\mathbb{C}} \ . \end{aligned} \tag{2.43}$$

$\mathrm{e}^{-\mathrm{i}J}$ and Ω_η are known as "pure spinors" [49] since their annihilator is a maximal isotropic subspace (for the case at hand, the annihilators of $e^{-\mathrm{i}J}$ and Ω_η are six-dimensional, and are given in [23, 28]). In terms of the $O(6,6)$ structure group on $F \oplus F^*$ each complex pure spinor defines a SU(3,3) sub-bundle. Together the common sub-bundle of the two SU(3,3) structures defined by $\mathrm{e}^{\mathrm{i}J}$ and Ω_η is a SU(3) $\times$ SU(3) bundle. Within this there is an SU(3) subgroup of diffeomorphisms $\mathrm{GL}(6,\mathbb{R}) \subset \mathrm{SO}(6,6)$ which leave J and ρ invariant and defines the original SU(3) structure. From this perspective, the SU(3) structure is defined by the pair of pure spinors $\mathrm{e}^{\mathrm{i}J}$ and Ω_η.

Now consider the case of a pair of SU(3) structures, given by η^A_+. As discussed above, this is the generic situation for the reformulated supergravity theory. Again one can construct two pairs of pure complex spinors $\eta^A_+ \otimes \bar{\eta}^A_+ \sqrt{\epsilon_g} = \mathrm{e}^{\mathrm{i}J^A}$ and $\eta^A_+ \otimes \bar{\eta}^A_- \sqrt{\epsilon_g} = \Omega^A_\eta$ with $A = 1,2$. It is actually more natural (and equivalent) to define, following [34, 50], the pure spinors

$$\begin{aligned}\Phi^+ &= \eta^1_+ \otimes \bar{\eta}^2_+ \sqrt{\epsilon_g} \in \Lambda^{\text{even}} F^*_{\mathbb{C}},\\ \Phi^- &= \eta^1_+ \otimes \bar{\eta}^2_- \sqrt{\epsilon_g} \in \Lambda^{\text{odd}} F^*_{\mathbb{C}}.\end{aligned} \tag{2.44}$$

Their expression in terms of the local $SU(2)$ structure is given in [34, 44]. Again here each pure spinor given in (2.44) defines a SU(3,3) structure. Since as Spin(6) spinors $\bar{\eta}^A_\pm \gamma_m \eta^A_+ = 0$, it is easy to show that $\bar{\Phi}^-_{\epsilon_g} \Gamma^\Sigma \Phi^+_{\epsilon_g} = 0$. This implies that together they define an SU(3) × SU(3) structure on $F \oplus F^*$. (This is discussed in more detail in section 2.4.6.) Writing the structure in terms of $\Phi^\pm$ the SU(3) × SU(3) action is immediately apparent: it corresponds to two independent SU(3) ⊂ Spin(6) groups acting separately on $\eta^1_\pm$ and $\eta^2_\pm$. In the generic case, globally there is no common subgroup of SU(3) × SU(3) and we simply have a pair of SU(3) structures. Locally there is a common SU(2) group, but this does not, generically, survive globally (this is the case when the spinors η^1 and η^2 become parallel at one or more points on the manifold.) Nonetheless, globally, the pair of SU(3) structures is equivalent to the pair of complex pure spinors given in eq. (2.44). In the special case where $\eta^1_+ = \eta^2_+ = \eta_+$, we have a single SU(3) structure with

$$\Phi^+ = \frac{1}{8} \mathrm{e}^{-\mathrm{i}J} , \qquad \Phi^- = -\frac{1}{8} \mathrm{i}\Omega_\eta . \tag{2.45}$$

This is the case with which we will be most concerned. However let us continue a little further in the more general setting of SU(3) × SU(3) structures and show that stable Spin(6,6) spinors define a special Kähler geometry following [51].

2.4.3 Special Kähler structure for stable Spin(6,6) spinors

Consider a general odd or even form χ, that is a section either of $\Lambda^{\text{even}} F^*$ or $\Lambda^{\text{odd}} F^*$. Let $\chi_\epsilon \in S^\pm$ be the corresponding spinor defined using the volume form ϵ. Following Hitchin [48, 51] we will show, first, that there is a natural special Kähler structure on the space of so called "stable" spinors χ_ϵ, and, second, how this also gives a special Kähler structure on the space of stable forms χ.

Just as a nowhere vanishing Spin(6) spinor defines an SU(3) structure, a nowhere vanishing Spin(6,6) spinor χ_ϵ defines an SU(3,3) structure. That is to say a generic χ_ϵ is invariant under SU(3,3) ⊂ Spin(6,6) rotations. As shown in [48], not quite all spinors define an SU(3,3) structure, but rather only an open subset of $S^\pm$ corresponding to so-called "stable spinors". To see how the stable spinors are defined it is useful to start by noting that, in analogy to the SU(3) case discussed in section 2.3.1, one can construct SU(3,3)-invariant forms out of the spinor bilinears. In particular, one can define a fundamental two-form $\mathcal{J} \in \Lambda^2(F \oplus F^*)$ given by

$$\mathcal{J}_{\Pi\Sigma} = \bar{\chi}_\epsilon \, \Gamma_{\Pi\Sigma} \, \chi_\epsilon , \qquad \Pi, \Sigma = 1, \dots, 12 , \tag{2.46}$$

where $\Gamma_{\Pi\Sigma}$ is the antisymmetrized product of two SO(6,6) Γ-matrices. Using the $O(6,6)$ metric one can raise one index forming $\mathcal{J}^\Pi{}_\Sigma$ which generically defines an almost complex structure. The only caveat is that $\mathcal{J}_{\Pi\Sigma}$ will not be properly normalized to be compatible with the SO(6,6) metric, that is $\mathcal{J}^\Pi{}_\Omega \mathcal{J}^\Omega{}_\Sigma = -k^2 \delta^\Pi{}_\Sigma$ but $k \neq 1$. This is because the normalization of χ_ϵ is not fixed, and there is no simple way to fix it, since $\bar{\chi}_\epsilon\chi_\epsilon = 0$ identically as can be seen, for example, from (2.36) or (2.38). Following Hitchin [51], one can instead introduce a quartic function of χ_ϵ, given by the square of $\mathcal{J}_{\Pi\Sigma}$. One defines

$$q_\epsilon(\chi_\epsilon) = -\frac{1}{4}(\bar{\chi}_\epsilon\Gamma_{\Pi\Sigma}\chi_\epsilon)(\bar{\chi}_\epsilon\Gamma^{\Pi\Sigma}\chi_\epsilon) \ , \tag{2.47}$$

together with the homogeneous Hitchin function of degree two

$$H_\epsilon(\chi_\epsilon) \ = \ \sqrt{-\frac{1}{3}\, q_\epsilon(\chi_\epsilon)} \ = \ \sqrt{\frac{1}{12}\,\mathcal{J}^2} \tag{2.48}$$

Note that when $\mathcal{J}$ is correctly normalized, $H_\epsilon(\chi_\epsilon) = 1$.

With the help of these functions one can define the notion of stable spinors. If $\mathcal{J}$ does define an almost complex structure then clearly $q_\epsilon(\chi_\epsilon) < 0$. In fact this is also sufficient for there to be an SU(3,3) structure. Hitchin defines the set of *stable* real Spin(6,6) spinors

$$U_\epsilon = \left\{\chi_\epsilon \in S^\pm : q_\epsilon(\chi_\epsilon) < 0\right\} \ . \tag{2.49}$$

By definition $q_\epsilon(\chi_\epsilon) \leq 0$, and hence one can see that U_ϵ is an open subset of $S^\pm$, consisting of all spinors such that $\mathcal{J} \neq 0$, and so is 32-dimensional. This is the statement that a generic spinor χ_ϵ defines a SU(3,3) structure. Clearly $k\chi^\pm$, for any non-zero $k \in \mathbb{R} - \{0\}$, defines the same structure. Hence the set of stable forms U_ϵ is a homogeneous space given by the set of SU(3,3) structures compatible with the $O(6,6)$ metric together with an overall scale [48]

$$U_\epsilon \simeq O(6,6) \otimes \mathbb{R}^+ / \, \mathrm{SU}(3,3). \tag{2.50}$$

In what follows it is useful to note that since $S^\pm$ is a vector space and U_ϵ is an open subset of $S^\pm$, there is a natural isomorphism between $T_{\chi_\epsilon}U_\epsilon$ at any point $\chi_\epsilon \in U_\epsilon$ and $S^\pm$. (For instance χ_ϵ can be viewed either as a coordinate on U_ϵ or the "position vector field" in TU_ϵ.)

On manifolds with SU(3) structure there is a pair of real invariant spinors, or equivalently a complex invariant spinor (and its complex conjugate). The same is true for SU(3,3) structures. The second real spinor $\hat{\chi}_\epsilon$ can be written in terms of χ_ϵ by acting with the correctly normalized operator $\mathcal{J}^{\Pi\Sigma}\Gamma_{\Pi\Sigma}$. Explicitly one has

$$\hat{\chi}_\epsilon = -\frac{1}{6\sqrt{\mathcal{J}^2/12}}\ \mathcal{J}^{\Pi\Sigma}\Gamma_{\Pi\Sigma}\,\chi_\epsilon \ . \tag{2.51}$$

This second spinor can also be defined in terms of the function $H_\epsilon(\chi_\epsilon)$. Recall that the spinor inner product defined a symplectic structure $\omega(\xi,\eta) = \bar{\xi}\eta$. The function $H_\epsilon(\chi_\epsilon)$ defines a Hamiltonian vector field $\hat{\chi}_\epsilon \in TU_\epsilon \simeq S^\pm$

$$i_{\hat{\chi}_\epsilon}\omega = -\mathrm{d}H_\epsilon \ , \tag{2.52}$$

where d is the exterior derivative on U_ϵ, so $\mathrm{d}H_\epsilon \in T^*U_\epsilon$. In components we have

$$\hat{\chi}^\alpha_\epsilon = -(\omega^{-1})^{\alpha\beta}\partial_\beta H_\epsilon(\chi_\epsilon), \tag{2.53}$$

where $\partial_\alpha = \partial/\partial\chi^\alpha_\epsilon$. Using (2.48) and (2.46) it is straightforward to see that (2.53) coincides with (2.51).

The corresponding complex spinor is

$$\Phi_\epsilon = \frac{1}{2}\left(\chi_\epsilon + \mathrm{i}\hat{\chi}_\epsilon\right), \tag{2.54}$$

which was shown to be a pure spinor in ref. [48]. In what follows it will be one of the two pure spinors defining the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure. As a vector field $\hat{\chi}_\epsilon \in TU_\epsilon$ generates the $U(1)$ action $\Phi_\epsilon \mapsto \mathrm{e}^{\mathrm{i}\theta}\Phi_\epsilon$ where we view the real and imaginary parts of Φ_ϵ as coordinates in U. Furthermore due to the homogeneity of H_ϵ together with the definition of $\hat{\chi}_\epsilon$ given in (2.52) and (2.53) one infers

$$H_\epsilon(\chi_\epsilon) = \frac{1}{2}\omega(\chi_\epsilon, \hat{\chi}_\epsilon) = \mathrm{i}\omega\big(\Phi_\epsilon, \bar{\Phi}_\epsilon\big) \ . \tag{2.55}$$

Now one can show that there is a natural rigid special Kähler structure on U. There is already a symplectic structure given by the spinor inner product. Since the matrix ω is constant, independent of χ_ϵ, we clearly have

$$\mathrm{d}\omega = 0. \tag{2.56}$$

Actually we have more. A constant ω gives a flat symplectic structure and implies that we can introduce Darboux coordinates $\chi_\epsilon^\alpha = (x^K, y_L)$ on U_ϵ with $K, L = 1, \ldots, 16$ such that

$$\omega = \mathrm{d}x^K \wedge \mathrm{d}y_K \ . \tag{2.57}$$

These will be useful in what follows.

Next one shows that the complex structure $\mathcal{J}$ on $F \oplus F^*$ also induces a complex structure I on TU_ϵ. Specifically, viewing $\hat{\chi}_\epsilon$ as an element of TU_ϵ, one defines $I \in TU \otimes T^*U$ by

$$I^\alpha{}_\beta = -\partial_\beta \hat{\chi}_\epsilon^\alpha = (\omega^{-1})^{\alpha\gamma}\partial_\gamma\partial_\beta H_\epsilon \ , \tag{2.58}$$

where the second equation follows from (2.53) since $\omega^{-1} = C$ is constant. In order to see that $I^2 = -\mathbf{1}$ we first note, from the definition (2.51), that $\bar{\hat{\chi}}_\epsilon \Gamma_{\Pi\Sigma}\hat{\chi}_\epsilon = \bar{\chi}_\epsilon \Gamma_{\Pi\Sigma}\chi_\epsilon$ and hence $H_\epsilon(\hat{\chi}_\epsilon) = H_\epsilon(\chi_\epsilon)$. Together with (2.55) this implies $\omega(\chi_\epsilon, \hat{\chi}_\epsilon) = \omega(\hat{\chi}_\epsilon, \hat{\hat{\chi}}_\epsilon)$ and hence

$$\hat{\hat{\chi}}_\epsilon = -\chi_\epsilon \ . \tag{2.59}$$

Taking a small variation of (2.53) and using (2.58) we find $\delta\hat{\chi}_\epsilon^\alpha = -I^\alpha{}_\beta \delta\chi_\epsilon^\beta$. Using this and (2.59) we find $\delta\chi_\epsilon^\alpha = -\delta\hat{\hat{\chi}}_\epsilon^\alpha = I^\alpha{}_\beta\, \delta\hat{\chi}_\epsilon^\beta = -I^\alpha{}_\beta\, I^\beta{}_\gamma\, \delta\chi_\epsilon^\gamma$ which indeed implies $I^2 = -\mathbf{1}$.

To show that I is also integrable one identifies explicit complex coordinates. In the Darboux coordinates (2.57) the complex spinor Φ_ϵ of (2.54) can be written as a complex vector on U_ϵ

$$\begin{aligned}
\Phi_\epsilon &= \frac{1}{2}\left(\chi_\epsilon + \mathrm{i}\hat{\chi}_\epsilon\right) \\
&= \frac{1}{2}\left(\chi_\epsilon^\alpha - \mathrm{i}(\omega^{-1})^{\alpha\beta}\partial_\beta H_\epsilon\right)\frac{\partial}{\partial\chi_\epsilon^\alpha} \\
&= \frac{1}{2}\left(x^K + \mathrm{i}\frac{\partial H_\epsilon}{\partial y_K}\right)\frac{\partial}{\partial x^K} + \frac{1}{2}\left(y_K - \mathrm{i}\frac{\partial H_\epsilon}{\partial x^K}\right)\frac{\partial}{\partial y_K} \\
&\equiv Z^K \frac{\partial}{\partial x^K} - F_K \frac{\partial}{\partial y_K} \ ,
\end{aligned} \tag{2.60}$$

where the last line defines the complex functions Z^K and F_K on U_ϵ. By definition the components $\Phi^\alpha_\epsilon = (Z^K, -F_K)$ satisfy $\mathrm{d}\Phi^\alpha_\epsilon = \frac{1}{2}\mathrm{d}\chi^\alpha_\epsilon - \frac{1}{2}\mathrm{i}I^\alpha{}_\beta \mathrm{d}\chi^\beta_\epsilon$. Hence $-\mathrm{i}(I\mathrm{d}\Phi_\epsilon)^\alpha = \mathrm{d}\Phi^\alpha_\epsilon$ and the one-forms $\mathrm{d}\Phi^\alpha_\epsilon = (\mathrm{d}Z^K, -\mathrm{d}F_K)$ are all of type $(1,0)$ with respect to I. This implies that Z^K and F_K are each separately complex coordinates on U. (They are known as conjugate coordinate systems.)

We have shown that on U_ϵ there exists a closed symplectic form ω and an integrable complex structure I. Furthermore, from (2.58), one sees that $\omega_{\alpha\gamma}I^\gamma{}_\beta = \partial_\alpha\partial_\beta H_\epsilon$ is symmetric. This implies ω and I are compatible (that is, ω is a $(1,1)$-form) and hence together they define a Kähler metric. The metric is given by

$$G_{\text{rigid}} = (\omega_{\alpha\gamma}I^\gamma{}_\beta)\, \mathrm{d}\chi^\alpha_\epsilon \otimes \mathrm{d}\chi^\beta_\epsilon = \partial_\alpha\partial_\beta H_\epsilon\, \mathrm{d}\chi^\alpha_\epsilon \otimes \mathrm{d}\chi^\beta_\epsilon\ . \tag{2.61}$$

If we change to complex coordinates Z^K, since G_{rigid} is by definition Hermitian, one has

$$G_{\text{rigid}} = \partial_K\bar\partial_L H_\epsilon\, \mathrm{d}Z^K \otimes \mathrm{d}\bar Z^L\ , \tag{2.62}$$

where $\partial_K = \partial/\partial Z^K$ and hence one can identify H_ϵ as the Kähler potential. Note that $\mathrm{d}x^K = \mathrm{d}Z^K + \mathrm{d}\bar Z^K$ and $\mathrm{d}y_K = \mathrm{d}F_K + \mathrm{d}\bar F_K$. It is then easy to see that the Hermitian condition (or equivalently the condition that ω is a $(1,1)$-form) implies that $\partial_{[K}F_{L]} = 0$. This implies that locally we can find a complex function F such that $F_K = \partial_K F$.

In summary, one sees, using (2.55) and (2.60), that the Kähler potential is given by

$$K_{\text{rigid}} \equiv H_\epsilon = \mathrm{i}\omega\big(\Phi_\epsilon, \bar\Phi_\epsilon\big) = \mathrm{i}\left(\bar Z^K F_K - Z^K \bar F_K\right), \tag{2.63}$$

where $F_K = \partial_K F$. Comparing with (2.26) and (2.27) we see that this is the standard form for a rigid special Kähler geometry.[6] One can further show that this is actually a pseudo-Kähler geometry: the signature of the metric G_{rigid} is $(30,2)$ [51].

Since we are interested in gravitational theories we really want to have a local special Kähler geometry. Fortunately there is a straightforward way of obtaining such a structure given the rigid geometry just described. Recall that $\hat\chi_\epsilon$ generated a $U(1)$ action on U_ϵ corresponding to $\Phi_\epsilon \to e^{\mathrm{i}\theta}\Phi_\epsilon$. The position vector field χ_ϵ generates a scaling $\Phi_\epsilon \to \lambda\Phi_\epsilon$. Together they define a $\mathbb{C}^*$ action compatible with the complex structure (since it is generated by the holomorphic vector field Φ_ϵ). Thus one can define the 30-dimensional quotient moduli space

$$\mathcal{M}_\epsilon = U_\epsilon/\mathbb{C}^*\ . \tag{2.64}$$

Under the $\mathbb{C}^*$ action the tangent space TU_ϵ descends to a holomorphic $\mathrm{Sp}(32,\mathbb{R})$ vector bundle $\mathcal{H}$ with symplectic structure ω. It is a standard result (see for instance [79]) that there is then a local special Kähler structure on $\mathcal{M}$ with Kähler potential

$$K = -\ln H_\epsilon = -\ln \mathrm{i}\omega\big(\Phi_\epsilon, \bar\Phi_\epsilon\big) = -\ln \mathrm{i}\left(\bar Z^K F_K - Z^K \bar F_K\right), \tag{2.65}$$

for some Φ_ϵ defined up to a Kähler transformation $\Phi_\epsilon \to \lambda\Phi_\epsilon$ where $\lambda \in \mathbb{C}^*$. (In [79] local special Kähler manifolds are defined as such quotients.) The corresponding metric is

[6] Anticipating the result we already denoted the holomorphic section of special Kähler geometry in section 2.4.1 by Φ.

Euclidean. The $Z^K(z)$ are complex sections of $\mathcal{H}$ where z are complex coordinates on $\mathcal{M}_\epsilon$. The moduli space corresponds to a space of $U(3,3)$ structures compatible with the $O(6,6)$ metric, so we can identify

$$\mathcal{M}_\epsilon \simeq O(6,6)/U(3,3), \tag{2.66}$$

as the space of complex structures compatible with the $O(6,6)$-metric. That $\mathcal{M}_\epsilon$ admits a Hermitian metric is well known (see for instance [46]). Hitchin's result [48] is to show that it has a natural special Kähler geometry.

Thus far the discussion has been in terms of the spinor χ_ϵ. As such, it appears that the special Kähler structures depend on the choice of volume form ϵ defining the isomorphism between forms and spinors. In fact, the final local special Kähler structure is actually independent of the choice of ϵ. All of the preceding discussion can be repeated in terms of the form $\chi \in \Lambda^{\text{even/odd}}F^*$. Explicitly, one defines the analog of $q_\epsilon(\chi_\epsilon)$, using the Mukai pairing (2.38) rather than the symplectic form ω,

$$q(\chi) = -\frac{1}{4}\left\langle \chi, \Gamma_{\Pi\Sigma}\chi\right\rangle \left\langle \chi, \Gamma^{\Pi\Sigma}\chi\right\rangle \ \in \Lambda^6 F^* \otimes \Lambda^6 F^* \ , \tag{2.67}$$

which is now formally the square of a volume form rather than a scalar. The open set of stable forms, which is isomorphic to U_ϵ, is then given by

$$U = \left\{ \chi \in \Lambda^{\text{even/odd}}F^* : q(\chi) < 0 \right\} \simeq U_\epsilon \ . \tag{2.68}$$

One has the analog of $H_\epsilon(\chi_\epsilon)$,

$$\begin{aligned} H(\chi) &= \sqrt{-\frac{1}{3}q(\chi)} \in \Lambda^6 F^* \\ &= H_\epsilon(\chi_\epsilon)\, \epsilon \end{aligned} \tag{2.69}$$

which is now a volume form.

The functional $H(\chi)$ defines a complex structure on U in complete analogy with the spinor case. One defines the Hamiltonian vector field $\hat{\chi}$ on $TU \simeq \Lambda^{\text{even/odd}}F^*$ by the action on TU

$$\langle \hat{\chi}, \cdot \rangle = -\mathrm{d}H(\cdot) \ , \tag{2.70}$$

and the corresponding complex vector field

$$\Phi = \frac{1}{2}\left(\chi + \mathrm{i}\hat{\chi}\right). \tag{2.71}$$

The complex structure is then defined as a derivative of Φ as before.

The difference arises in the symplectic structure. We can define a map $\Lambda^{\text{even/odd}}F^* \otimes \Lambda^{\text{even/odd}}F^* \to \mathbb{R}$ by $(\chi, \psi) \mapsto \omega(\chi_\epsilon, \psi_\epsilon)$ but this depends on the choice of ϵ. This means that there is no canonical rigid special Kähler metric on U, but only a family of metrics depending on ϵ with Kähler potentials $K_{\text{rigid}} = H_\epsilon(\chi_\epsilon)$. However, we can again form the quotient moduli space

$$\mathcal{M} = U/\mathbb{C}^* \simeq \mathcal{M}_\epsilon \ . \tag{2.72}$$

The corresponding local special Kähler potential given by $K = -\ln H_\epsilon$ is *independent* of the particular choice of ϵ in defining the symplectic structure, since rescaling ϵ simply shifts K by a constant and corresponds to a Kähler transformation. From this perspective we can introduce the volume form

$$e^{-K} = H = i \left\langle \Phi, \bar{\Phi} \right\rangle = i(\bar{Z}^K F_K - Z^K \bar{F}_K) \in \Lambda^6 F^*, \tag{2.73}$$

where we have introduced complex homogeneous coordinates $Z^K(z)$ as above and z^a are complex coordinates on $\mathcal{M}$. The special Kähler metric is then given by

$$ds^2 = \left(\frac{\partial_a \bar{\partial}_b H}{H} - \frac{\partial_a H}{H} \frac{\bar{\partial}_a H}{H} \right) \delta z^a \delta \bar{z}^b, \tag{2.74}$$

where $\partial_a = \partial / \partial z^a$ and the powers of $\Lambda^6 F^*$ cancel so that the metric G really is a map $G : \Lambda^{\text{even/odd}} F^* \otimes \Lambda^{\text{even/odd}} F^* \to \mathbb{R}$.

In summary, there is a unique special Kähler structure on the (quotient) space $\mathcal{M} = U/\mathbb{C}^*$ of stable forms $\chi \in \Lambda^{\text{even/odd}} F^*$ where the exponentiated Kähler potential e^{-K} is naturally a six-form given by the Hitchin function $H(\chi)$. From now on we will consider only this structure on the space of forms and not consider the corresponding spinors.

2.4.4 Special Kähler structure for stable J

Having discussed the general case, let us now turn to the specific special Kähler structures that arise from ρ and J. Let us start with the symplectic two-form J. From Calabi-Yau compactifications we know that it is naturally paired with the NS two-form B. To match with the discussion of the previous section, we introduce the Spin(6, 6) spinor

$$\chi^+ = 2 \, \mathrm{Re}(c \, e^{-B - iJ}) \in \Lambda^{\text{even}} F^* \tag{2.75}$$

where $c \in \mathbb{C} - \{0\}$. Note that for general J, B and c the spinor χ^+ is completely generic.

Using Hitchin's construction we find by inserting (2.75) into (2.69) using (2.32)

$$H(\chi^+) = \frac{4}{3} |c|^2 J \wedge J \wedge J \ . \tag{2.76}$$

(One way to see this is to note that the phase of c and the entire B dependence can be removed by a $O(6, 6)$ rotation and hence both drop out of $H(\chi)$.) The space of stable χ^+ is then given by

$$U_J = \left\{ \mathrm{Re}(c \, e^{-B - iJ}) \in \Lambda^{\text{even}} F^* : J \wedge J \wedge J \neq 0 \right\} \ . \tag{2.77}$$

This is equivalent to the usual condition that J is non-degenerate. The second spinor $\hat{\chi}$ is found from (2.51) to be $2 \, \mathrm{Im}(c \, e^{-B - iJ})$ and thus the pure complex spinor reads

$$\Phi^+ = c \, e^{-B - iJ} \ . \tag{2.78}$$

From (2.38) we see that $\Lambda^0 F^* \oplus \Lambda^2 F^*$ forms a maximal null subspace of $\Lambda^{\text{even}} F^*$ under the inner product $\langle \cdot, \cdot \rangle$. Thus we can choose symplectic Darboux coordinates such that the x^A span $\Lambda^0 F^* \oplus \Lambda^2 F^*$. In order to distinguish from the special Kähler structure for ρ which we will discuss in the next section let us denote the complex coordinates on U_J by X^A and $\mathcal{F}_A$ (instead of Z^K and F_K). Expanding Φ^+ we determine the X^A to be

$$X^0 = c, \qquad X^a = -c(B + \mathrm{i}J)^a \tag{2.79}$$

where $a = [mn]$ running from 1 to 15 denotes the pair of antisymmetric indices on the two forms. Finally, the $\mathbb{C}^*$ action generated by Φ^+ acts by rescaling c, that is $c \to \lambda c$ with $\lambda \in \mathbb{C}^*$.

From Hitchin's construction we immediately obtain a special Kähler manifold on the quotient space

$$\mathcal{M}_J = U_J / \mathbb{C}^* \ , \tag{2.80}$$

with the Kähler potential K_J given by

$$\mathrm{e}^{-K_J} = H = \frac{4}{3} |c|^2 \ J \wedge J \wedge J \ , \tag{2.81}$$

and we see e^{-K} is naturally a six-form as discussed above. On $\mathcal{M}_J$ one can introduce special coordinates defined in section 2.4.1

$$t^a = -X^a / X^0 = (B + \mathrm{i}J)^a \ . \tag{2.82}$$

(Note we have introduced an extra sign to match the conventional definition of t^a.) In these coordinates the prepotential $f(t^a)$ introduced in (2.29) then takes the form

$$\mathcal{F}(X^A) = (X^0)^2 f(t^a) \ , \qquad f(t^a)\, \epsilon = t \wedge t \wedge t \ . \tag{2.83}$$

Note that these expressions are exactly analogous to the expressions for the special Kähler geometry on the space of Kähler deformations on a Calabi-Yau manifold except that here J and B are functions of all ten spacetime coordinates and we do not necessarily have a Calabi-Yau manifold.

Finally we must show that the metric defined by K_J corresponds to the metric for the supergravity kinetic terms $\mathrm{d}s^2(J, B)$ given in (2.25). Starting from the Kähler potential (2.81) we find the Kähler metric

$$\mathrm{d}s_J^2 = -\frac{3}{2} \left[\frac{\delta t \wedge \delta \bar{t} \wedge J}{J^3} - \frac{3}{2} \frac{\delta t \wedge J^2}{J^3} \frac{\delta \bar{t} \wedge J^2}{J^3} \right] \ . \tag{2.84}$$

Note that here we see explicitly that both numerator and denominator are proportional to the volume form which therefore cancels in the ratio. Rewriting this expression in terms of contractions with the metric g_{mn} we have

$$\mathrm{d}s_J^2 = \frac{3}{4} \delta\lambda \delta\lambda + \frac{1}{2} g^{mn} \delta v_m \delta v_n + \frac{1}{8} g^{mp} g^{nq} \delta K_{mn} \delta K_{pq} + \frac{1}{8} g^{mp} g^{nq} \delta B_{mn} \delta B_{pq}, \tag{2.85}$$

which precisely matches the metric (2.25) (up to the terms involving the vector deformation δv). Again this is similar to the situation in Calabi-Yau manifolds where the analogous computation can be found in refs. [5, 7]. Recall that the δv terms do not correspond to physical deformations but are associated with different SU(3) structures which define the same metric. In the full theory we expect them not to be present. We discuss this in some more detail in section 2.4.6. In the limit where we drop additional spin-$\frac{3}{2}$ multiplets and hence all fields in the $\mathbf{3}+\bar{\mathbf{3}}$ representation of SU(3), we set $\delta v = 0$ and the agreement is exact.

2.4.5 Special Kähler structure for stable ρ

We now turn to the almost complex structure on F defined by $\rho \in \Lambda^3 F^*$. Unlike the previous case this is not a generic Spin(6,6) spinor since it does not contain one-form or five-form pieces. Nonetheless an analogous construction of the special Kähler moduli space exists. (This was also first given by Hitchin in [51].)

One starts by noting that the pairing $\langle \cdot, \cdot \rangle$ defined in (2.38) still defines a symplectic structure on the subspace $\Lambda^3 F^* \in \Lambda^{\text{odd}} F^*$. Then one introduces the (restricted) spinor

$$\chi^- = 2n\,\rho \in \Lambda^3 F^* \subset \Lambda^{\text{odd}} F^* \ , \tag{2.86}$$

where the factor of two is for later convenience and $n \in \mathbb{R}-\{0\}$ is an arbitrary normalization constant analogous to the c introduced in (2.75). For this restricted spinor one can still define the forms $q(\chi^-)$ and $H(\chi^-)$ as in (2.67) and (2.71). Explicitly one finds, using (2.32),

$$q(\chi^-) = 8\,n^4\,(e^m \wedge i_n\rho \wedge \rho)(e^n \wedge i_m\rho \wedge \rho) \ \in \ \Lambda^6 F^* \otimes \Lambda^6 F^* \ . \tag{2.87}$$

The space of stable three-forms is

$$U_\rho = \left\{\rho \in \Lambda^3 F^* : q(\rho) < 0\right\} \ . \tag{2.88}$$

The key here is that only the $GL(6,\mathbb{R}) \subset \text{Spin}(6,6)$ generators in $\Gamma^{\Pi\Sigma}$ give a non-zero contribution to $q(\rho)$. From this perspective, rather than defining a SU(3,3) substructure of Spin(6,6), the three-form ρ defines a $SL(3,\mathbb{C})$ substructure of $GL(6,\mathbb{R})$. The spinor stability then reduces to a notion of three-form stability, that is U_ρ defines an open orbit under $GL(6,\mathbb{R})$. Correspondingly U_ρ is isomorphic to the homogeneous space

$$U_\rho \simeq GL^+(6,\mathbb{R})/\,SL(3,\mathbb{C}), \tag{2.89}$$

where $GL^+(6,\mathbb{R})$ is the space of real matrices with positive determinant. Note if we had considered the generic case of $SU(3)\times SU(3)$ structures, rather than restricting to SU(3) structures, χ^- as defined (2.44) would be a generic element of $\Lambda^{\text{odd}} F^*$ and we would be back to the general Hitchin construction.

Given $H(\chi^-)$ the construction continues just as above. One defines $\hat{\chi}^- \in \Lambda^3 F^*$, by

$$\left\langle \hat{\chi}^-, \cdot \right\rangle = -\mathrm{d}H(\cdot) \tag{2.90}$$

so $\hat{\chi}^- = 2n\hat{\rho}$ and then the complex spinor

$$\Omega = \chi^- + \mathrm{i}\hat{\chi}^- = n\Omega_\eta \in \Lambda^3 F^*_{\mathbb{C}} \ , \tag{2.91}$$

which is the $(3,0)$-form usually used to define a SU(3) structure. (Recall that Ω_η was defined in (2.12) in terms of normalized spinors $\bar{\eta}\eta = 1$ and Ω differs from Ω_η by the arbitrary normalization n.)

The $\mathbb{C}^*$ action generated by Ω acts as $\Omega \to \lambda\Omega$ with $\lambda \in \mathbb{C}^*$. One then constructs the quotient space

$$\mathcal{M}_\rho = U_\rho/\mathbb{C}^* \ , \tag{2.92}$$

with the Kähler potential K_ρ is defined via the six-form

$$\mathrm{e}^{-K_\rho} = H = \mathrm{i}\Omega \wedge \bar{\Omega} \ . \tag{2.93}$$

Note that modding out by the $\mathbb{C}^*$ action means that points in $\mathcal{M}_\rho$ do not distinguish the phase of Ω. Thus $\mathcal{M}_\rho$ is the moduli space of almost complex structures rather than $\mathrm{SL}(3,\mathbb{C})$ structures on F and we can identify

$$\mathcal{M}_\rho \simeq \mathrm{GL}^+(6,\mathbb{R})/\,\mathrm{GL}^+(3,\mathbb{C}). \tag{2.94}$$

Choosing a symplectic basis of $\Lambda^3 F^*$ one can introduce complex coordinates Z^K. Crucially, as above, Ω is a holomorphic section, that is it depends only on the holomorphic coordinates z^k on $\mathcal{M}_\rho$.

As for the case with J, one notes that the special Kähler geometry is governed by expressions which are exactly analogous to the corresponding expressions for the special Kähler geometry of complex structure deformations on a Calabi-Yau manifold [7]. The difference here is again that Ω is a function of all ten spacetime coordinates and we do not necessarily have a Calabi-Yau manifold.

Finally we must show that the metric defined by K_ρ indeed coincides with the scalar kinetic term metric $\mathrm{d}s^2(\rho)$ given in (2.25). To do so it is convenient to note that, using the Mukai pairing, any complex three-form can be decomposed into a part along Ω and an orthogonal piece. In particular, for the derivative $\partial\Omega/\partial z^k$ where z^k are holomorphic coordinates on $\mathcal{M}$, we can always write

$$\frac{\partial\Omega}{\partial z^k} = K_k\Omega + \psi_k \tag{2.95}$$

where ψ_k is defined by $\langle \psi_k, \bar{\Omega} \rangle = 0$. The metric on $\mathcal{M}$ then takes the form

$$\mathrm{d}s^2_\rho = -\frac{\mathrm{i}\psi_k \delta z^k \wedge \bar{\psi}_k \delta \bar{z}^k}{\mathrm{i}\Omega \wedge \bar{\Omega}}. \tag{2.96}$$

Given the expansion of ρ in (2.17) we can identify

$$K_k \delta z^k = \frac{3}{2}\delta\lambda - \mathrm{i}\delta\gamma, \qquad \psi_k \delta z^k = (\delta M - \delta v \wedge J) + \mathrm{i} * (\delta M + \delta v \wedge J)\,. \tag{2.97}$$

Substituting into (2.96) and rewriting in terms of contractions with g_{mn} one finds

$$\mathrm{d}s^2_\rho = \frac{1}{24} g^{mr} g^{ns} g^{pt} \delta M_{mnp} \delta M_{rst} + \frac{1}{2} g^{mn} \delta v_m \delta v_n \ , \tag{2.98}$$

which exactly matches the scalar kinetic term metric of (2.25) (up to terms in the vector deformation δv). Again this is analogous to the Calabi-Yau computation which can be found in ref. [7]. As before the δv deformation is dropped when we remove the extra spin-$\frac{3}{2}$ multiplets by ignoring all fields in the $\mathbf{3} + \bar{\mathbf{3}}$ representation of SU(3).

2.4.6 Special Kähler structure for metric $SU(3) \times SU(3)$-structures

In the previous two sections we have seen that there are natural special Kähler structures on the space of two forms $B + iJ$ and three-forms ρ respectively, each of which agrees with the corresponding metric in the type II supergravity theory. However rather than general J and ρ the actual supergravity degrees of freedom are the metric g. In this section, we discuss some of the issues involved in removing the redundant degrees of freedom. However a full analysis of the structures which arise will depend on including the couplings of the additional spin-$\frac{3}{2}$ multiplets and hence our discussion will not be complete.

To go from J and ρ to g involves two steps. First one imposes the conditions

$$J \wedge \rho = 0, \qquad J^3 = \frac{3}{2}\rho \wedge \hat{\rho}. \tag{2.99}$$

These imply that together J and ρ define an SU(3)-structure. Secondly we need to mod out by those degrees of freedom that are not physical, that is take equivalence classes of (J, ρ) which describe the same metric g on F. In terms of deformations these are the degrees of freedom parameterized by v and γ in (2.17). Given any stable ρ we can always rescale ρ so as to satisfy the second condition in (2.99). However modding out by the scale of ρ together with the γ deformation corresponds precisely to the $\mathbb{C}^*$ action on U_ρ used to define the moduli space $\mathcal{M}_\rho$. Similarly the $\mathbb{C}^*$ action on U_J removes the non-physical degree of freedom c in (2.75). Thus the actual problem on $\mathcal{M}_J \times \mathcal{M}_\rho$ is to impose the first condition in (2.99) and mod out by the vector v degrees of freedom.

Crucially, one notes that both the constraint $J \wedge \rho = 0$ and v transform in the $\mathbf{3} + \bar{\mathbf{3}}$ representation under SU(3). In the approximation where we are dropping the additional spin-$\frac{3}{2}$ multiplets we drop all triplet representations. Thus, in this case, the condition is necessarily satisfied, there are no v deformations and (J, ρ) define a unique metric. This is precisely what happens for Calabi-Yau manifolds and also will be our assumption in section 3 where we will consider truncations of the general theory. Here we will discuss something of what form the general theory may take.

Let us start with the constraint. In fact, it is straightforward to consider the general case of $SU(3) \times SU(3)$ structures. Recall that a generic stable form $\chi^\pm \in \Lambda^{\text{even/odd}}$ defines an $SU(3,3)$ structure. Let $U^\pm$ denote the corresponding spaces of stable $\chi^\pm$, and $\mathcal{M}^\pm = U^\pm/\mathbb{C}^*$. We must identify those pairs of spinors (χ^+, χ^-) which determine an $SU(3) \times SU(3)$ structure, as defined in ref. [34]. (In the case where the structure is integrable, this should presumably be equivalent to what Gualtieri [49] calls a "generalized Calabi-Yau metric".) This condition implies that the spinors define a metric g and B-field. Taking a generic pair is not sufficient: we must impose a condition. This is the generalization of the conditions (2.99) that J and ρ together define a metric g. It defines a subspace

$$V \hookrightarrow U^+ \times U^-, \tag{2.100}$$

which can be identified with the 52-dimensional homogeneous space

$$V \simeq O(6,6) \otimes \mathbb{R}^+ \otimes \mathbb{R}^+/(SU(3) \otimes SU(3)). \tag{2.101}$$

The two factors of $\mathbb{R}^+$ here correspond to the fact that rescaling either $\chi^\pm$ by a non-zero real number defines the same $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure.

The condition can be expressed in a number of ways. Let $\mathcal{J}^{\pm\,\Pi}{}_\Sigma$ be the pair of almost complex structures (2.46) on $F \oplus F^*$ defined by $\chi^\pm_\epsilon$. Two equivalent formulations of the condition are

$$[\mathcal{J}^+, \mathcal{J}^-] = 0 \quad \Leftrightarrow \quad \left\langle \chi^+, \Gamma^\Pi \chi^- \right\rangle = 0. \tag{2.102}$$

The first form was first given in [49].[7] To see that the second form is a necessary condition one simply notes that there are no scalars in the decomposition of the vector representation of $O(6,6)$ under $\mathrm{SU}(3) \times \mathrm{SU}(3)$. From it we learn that V is a 52-dimensional subspace of $U^+ \times U^-$. To see explicitly how that metric and B-field arise, one defines a metric $\mathcal{G}$ on $F \oplus F^*$ as

$$\mathcal{G} = \frac{\mathcal{J}^+ \mathcal{J}^-}{\sqrt{H_\epsilon(\chi^+_\epsilon) H_\epsilon(\chi^-_\epsilon)}} = \frac{\mathcal{J}^- \mathcal{J}^+}{\sqrt{H_\epsilon(\chi^+_\epsilon) H_\epsilon(\chi^-_\epsilon)}} \,. \tag{2.103}$$

The factors of $H_\epsilon(\chi^\pm_\epsilon)$ normalize the almost complex structures, and also ensure that $\mathcal{G}$ is independent of ϵ. The metric can be written in terms of g and B as [49]

$$\mathcal{G}^\Pi{}_\Sigma = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}, \tag{2.104}$$

a form familiar from discussions of T-duality [77]. If we restrict to the special case of $\mathrm{SU}(3)$ structures, the condition (2.102) is equivalent to

$$J \wedge \rho = B \wedge \rho = 0 \tag{2.105}$$

implying that when restricting to $\mathrm{SU}(3)$ structures it is natural to restrict B as well.

The condition (2.102) can also be written in terms of the complex (pure) spinors $\Phi^\pm$ as

$$\left\langle \Phi^+, \Gamma^\Pi \Phi^- \right\rangle = 0. \tag{2.106}$$

(For $\mathrm{SU}(3)$ structures the corresponding condition is $(B + \mathrm{i}J) \wedge \Omega = 0$.) This form is manifestly invariant under the $\mathbb{C}^*$ actions on $U^\pm$ used to define the moduli spaces $\mathcal{M}^\pm$. Thus we can also view the condition as defining a subspace of $\mathcal{M}^+ \times \mathcal{M}^-$

$$\mathcal{N} \hookrightarrow \mathcal{M}^+ \times \mathcal{M}^-. \tag{2.107}$$

Equivalently we can think of $\mathcal{N}$ as the quotient of V by the two $\mathbb{C}^*$ actions. Again this gives a homogeneous space, now 48-dimensional,

$$\mathcal{N} \simeq O(6,6)/(U(3) \times U(3)). \tag{2.108}$$

Since $\Phi^\pm$ are holomorphic functions on $\mathcal{M}^\pm$ the condition (2.106) represents a complex submanifold $\mathcal{N}$. This means that the special Kähler structure on the total space $\mathcal{M}^+ \times \mathcal{M}^-$ induces a special Kähler structure on $\mathcal{N}$. Crucially however $\mathcal{N}$ is no longer a product

[7] We thank Marco Gualtieri for discussions confirming the second form of the condition.

manifold where the χ^+ and χ^- structures separate into one special Kähler manifold for the vector multiplet scalars and one for (part of) the hypermultiplet scalars. This is contrary to the usual $d = 4, N = 2$ form of supergravity.

Even on the constrained manifold $\mathcal{N}$ we are over-counting the degrees of freedom since many $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structures define the same metric and B-field. As we already observed in (2.25) the actual 36-dimensional space of g and B is isomorphic to the homogeneous Narain moduli space [77]

$$\mathcal{Q} = O(6,6)/(O(6) \times O(6)). \tag{2.109}$$

This can be obtained from $\mathcal{N}$ by modding out by those elements of $O(6) \times O(6)$ not in $\mathrm{SU}(3) \times \mathrm{SU}(3)$. In the $\mathrm{SU}(3)$ case this corresponds to modding out by the v deformations. This implies that $\mathcal{N}$ is a fibration over $\mathcal{Q}$

$$\begin{array}{ccc} \mathbb{C}P^3 \times \mathbb{C}P^3 & \longrightarrow & \mathcal{N} \\ & & \Big\downarrow \\ & & \mathcal{Q} \end{array} \tag{2.110}$$

with fibers

$$O(6)/U(3) \times O(6)/U(3) \simeq \mathbb{C}P^3 \times \mathbb{C}P^3. \tag{2.111}$$

Although the fibers admit a complex structure, it appears that with respect to the special Kähler metric on $\mathcal{N}$ this is not a complex fibration, and so there is no natural complex structure or Kähler metric on $\mathcal{Q}$. Note that in the restricted case of $\mathrm{SU}(3)$ structures the corresponding fibration is by a single $\mathbb{C}P^3$ factor.

We see that in the full theory reducing to the physical degrees of freedom in g and B takes us away from the usual form of $N = 2$ supergravity: first since $\mathcal{N}$ is not a product, and second in that there is no natural special Kähler structure on $\mathcal{Q}$. Of course this is related to the fact that keeping the $SU(3)$ triplets correspond to additional spin-$\frac{3}{2}$ gravitinos and additional vectors and scalars which enlarge the $N = 2$ multiplets to multiplets of higher N. More precisely, if we also include all scalar fields from the RR-sector the combined field space would be the $N = 8$ coset $E_{7(7)}/SU(8)$. Here we concentrated only on the metric and B-field deformations and hence only discovered the NS-subspace $\mathcal{Q}$ given in (2.109).

In summary, we have seen from Hitchin's results that there are natural special Kähler metrics on the spaces of $\mathrm{SU}(3,3)$ structures given by spinors $\eta^\pm$. In the special case of $\mathrm{SU}(3)$ structures (J,ρ) together with the two-form field B, we have the Kähler potentials

$$\begin{aligned} \mathrm{e}^{-K_J} &= \frac{4}{3}|c|^2 J \wedge J \wedge J \ , \\ \mathrm{e}^{-K_\rho} &= \mathrm{i}\Omega \wedge \bar{\Omega} \ . \end{aligned} \tag{2.112}$$

These are the obvious generalizations of the corresponding Calabi-Yau Kähler potentials and correspond to the Hitchin functionals $H(\chi^\pm)$ where $\chi^+ = \mathrm{Re}(2c\,\mathrm{e}^{-B-\mathrm{i}J})$ and $\chi^- = n\rho$. The construction generalizes to $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures simply by taking χ^- to be a generic (stable) odd-form, this is, with one-form and five-form pieces in addition to ρ. If we ignore all triplet representations these Kähler potentials give the metrics calculated

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directly from supergravity. In addition all the degrees of freedom in J and ρ are physical. If we drop this condition we must impose the SU(3) structure constraints (2.105) and mod out by the non-physical deformation v. The resulting structures are not naturally special Kähler manifolds but this is not surprising since in this case we must include additional multiplets which modify the scalar field sector.

2.5 Supersymmetry transformation of the gravitinos

So far we concentrated on the moduli space $\mathcal{M}$ of metric deformations and showed that it can be universally determined independently of the specific class of $SU(3)$ structure under consideration. We have seen that the "kinetic terms" in the type II supergravity are realized in terms of a special Kähler sigma model metric on $\mathcal{M}$ with a Kähler potential given in terms of the corresponding Hitchin functionals. Let us now turn to the contributions of the "potential terms" that is terms without derivatives in the $T^{1,3}$ bundle.

In supersymmetric theories the potential is given generically by the sum of the squares of the scalar part of the supersymmetry tranformations of the fermion fields. Furthermore for the fermions in the vector, tensor and hypermultiplets this scalar part of the variations is determined by derivatives of the scalar part of the gravitino variation [74]. Or in other words the transformation of the gravitino is the fundamental quantity from which one obtains all terms in the potential by appropriate derivatives.

In four-dimensional $N = 2$ supergravity the transformation of the gravitinos has the generic form

$$\delta\psi_{A\,\mu} = D_\mu \varepsilon_A + i\gamma_\mu S_{AB}\varepsilon^B \ , \tag{2.113}$$

where ε_A with $A = 1, 2$ are the two supersymmetry parameters of $N = 2$, while ε^A are the conjugate spinors . In terms of the ten-dimensional decomposition (2.3), we have $\varepsilon_A = \varepsilon_+^A$ and $\varepsilon^A = \varepsilon_-^A$. S_{AB} is an $SU(2)$ matrix[8] which can equivalently be expressed in terms of three Killing prepotentials $\mathcal{P}^x, x = 1, 2, 3$ via

$$S_{AB} = \frac{\mathrm{i}}{2}\,\mathrm{e}^{\frac{1}{2}K_V}\sigma^x_{AB}\mathcal{P}^x, \qquad \sigma^x_{AB} = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix}, \tag{2.114}$$

where K_V is the Kähler potential of the vector multiplets. The three $\mathcal{P}^x$ can be viewed as the $N = 2$ equivalent of the $N = 1$ superpotential and the $N = 1$ D-term. Together with its derivatives, S (or $\mathcal{P}^x$) determines the scalar potential.

In the spirit of this section we now want to 'lift' this discussion to the full ten-dimensional theory in a background with $SU(3)$ structure. To do so we simply compute the supersymmetry transformation of the gravitinos which reside in the gravitational multiplet and write it in a form analogous to (2.113). From the result we then read off the ten-dimensional analogue of S_{AB} or $\mathcal{P}^x$.

[8] The four-dimensional $N = 2$ theory has a local $SU(2)_R$ symmetry which rotates the two (complex) gravitinos $\psi_{A\,\mu}$ into each other. In ten dimensions it arises from the $O(2)$ rotation of the two ten-dimensional Majorana-Weyl fermions into each other together with additional generators which, as we will see, arise in the decomposition (2.117), (2.118). This $SU(2)_R$ is unrelated to the $SU(3)$ or $SU(3) \times SU(3)$ discussed so far in this section.

In tables 4, 5 and 6 we determined that the gravitinos in the gravitational multiplet can be easily identified as the SU(3)-singlet of Ψ_μ. However, the supersymmetry transformation (2.113) in addition requires that its kinetic term is diagonal and no mixed terms with the spin-$\frac{1}{2}$ fermions occur. This determines the gravitino as a particular linear combination among the fermions which now determine.

We start from the canonical ten-dimensional kinetic term for Ψ_M. Here it is actually most convenient to start in the ten-dimensional Einstein frame with metric g_E related to the string frame metric g by $g_E = \mathrm{e}^{-\phi/2} g$, since the conventional gravitino field in this frame has no derivative coupling to the dilatino field. The kinetic term then takes the form

$$S_{\text{gravitino}} = -\frac{1}{\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g_E}\, \bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P \ , \qquad M,N,\ldots = 0,\ldots,9 \ . \tag{2.115}$$

One can immediately see that the simple split $\Psi_M = (\Psi_\mu, \Psi_m)$ does not lead to diagonal kinetic terms. Instead we have to redefine the gravitino according to

$$\hat{\Psi}_\mu \equiv \Psi_\mu + \frac{1}{2}\Gamma_\mu{}^m \Psi_m \ , \tag{2.116}$$

where $\Gamma_\mu{}^m = g_E^{mn}\Gamma_{[\mu m]}$. When inserted into (2.115) one easily checks that $\hat{\Psi}_\mu$ now has a diagonal kinetic term and therefore should be identified as the field which transforms as in (2.113).

Using (2.11) and the fact that we need to focus on the singlet part of $\hat{\Psi}_\mu$ we can express $\hat{\Psi}_\mu$ in terms of the SU(3) singlet spinors $\eta_\pm$. For type IIA we thus have

$$\begin{aligned} \hat{\Psi}^{\text{IIA}}_{1\,\mu} &= \psi_{1\,\mu+} \otimes \eta_+ + \psi_{1\,\mu-} \otimes \eta_- + \ldots \ , \\ \hat{\Psi}^{\text{IIA}}_{2\,\mu} &= \psi_{2\,\mu-} \otimes \eta_+ + \psi_{2\,\mu+} \otimes \eta_- + \ldots \ , \end{aligned} \tag{2.117}$$

where we omitted the triplets. The indices 1 and 2 distinguish the two gravitinos which have opposite chirality in IIA. By slight abuse of notation we use plus and minus to indicate both four-dimensonal and six-dimensional chiralities, respectively. In type IIB both gravitinos have the same chirality (which we take to be negative) and thus the appropriate decomposition reads

$$\hat{\Psi}^{\text{IIB}}_{A\,\mu} = \psi_{A\,\mu+} \otimes \eta_- + \psi_{A\,\mu-} \otimes \eta_+ + \ldots \ , \qquad A = 1,2 \ . \tag{2.118}$$

Recall that there is a similar decomposition of the supersymmetry parameters given in eqs. (2.3) and (2.4). For simplicity let us consider in the follwing the special case where we have a SU(3) structure rather than a SU(3)×SU(3) structure and thus take $\eta^1_+ = \eta^2_+ = \eta_+$. (The computation for a more general SU(3)×SU(3) structure will be presented elsewhere.)

In type II supergravity the supersymmetry transformation of the gravitinos in the Einstein frame is

$$\begin{aligned} \delta\Psi_M = {} & D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\phi/2}\left(\Gamma_M{}^{PQR}H_{PQR} - 9\Gamma^{PQ}H_{MPQ}\right)P\epsilon \\ & - \sum_n \frac{\mathrm{e}^{(5-n)\phi/4}}{64\,n!}\left[(n-1)\Gamma_M{}^{N_1\ldots N_n} - n(9-n)\delta_M{}^{N_1}\Gamma^{N_2\ldots N_n}\right] F_{N_1\ldots N_n}\, P_n\, \epsilon \, , \end{aligned} \tag{2.119}$$

We are using the democratic formulation of [80], for which the sum is over $n = 0, 2, 4, 6, 8$, $P = \Gamma_{11}$ and $P_n = -(\Gamma_{11})^{n/2}\sigma^1$ for IIA. For IIB we have instead sum over $n = 1, 3, 5, 7, 9$, $P = -\sigma^3$ and $P_n = i\sigma^2$ for $n = 1, 5, 9$ and $P_n = \sigma^1$ for $n = 3, 7$. Furthermore,

$$F_n = dC_{n-1} - H \wedge C_{n-3} \tag{2.120}$$

are the modified RR field strengths with non standard Bianchi identities.

Turning on only fluxes that preserve Poincare invariance, and using the duality relation $F_n = (-1)^{\mathrm{Int}[n/2]} * F_{10-n}$, we can write the supersymmetry transformation in terms of only internal fluxes F_n, $n = 0, \ldots, 6$. For instance a non-zero F_4 with only μ-type indices is traded for a "internal" F_6 with m-type indices. In (2.119) this gives twice the contribution for each flux but now n only takes the values $n = 0, \ldots, 6$.

To extract the supersymmetry transformation of ψ_μ^A we need to project onto the SU(3)-singlet part of the variation of the shifted gravitino $\hat{\Psi}_\mu^A$. Given the normalizations of $\eta_\pm$ the relevant projection operators are

$$\Pi_\pm = \mathbf{1} \otimes 2\,(\eta_\pm \otimes \bar{\eta}_\pm) \ . \tag{2.121}$$

In order to extract the supersymmetry transformation of $\psi_{\mu+}$, we should use Π_+ (or Π_-) when dealing with a 10-dimensional positive (or negative) chirality spinor. To match our definitions in the previous section, we will also rewrite the variation in terms of the string frame metric $g = \mathrm{e}^{\phi/2} g_E$ in the following. This introduces additional factors of the dilaton into the expressions in (2.119).

Let us first focus on type IIB for which we evaluate, in the string frame,

$$\begin{aligned}
\Pi_-\,\delta\hat{\Psi}_\mu &= D_\mu\varepsilon_+ \otimes \eta_- - \frac{1}{48}\left(\bar{\eta}_-\gamma^{mnp}\eta_+ H_{mnp}\right)\gamma_\mu\sigma^3\epsilon_- \otimes \eta_- \\
&\qquad\qquad - \frac{1}{48}\left(\mathrm{e}^\phi\,\bar{\eta}_-\gamma^{mnp}\eta_+ F_{mnp}\right)\gamma_\mu\sigma^1\epsilon_- \otimes \eta_- \\
&= D_\mu\varepsilon_+ \otimes \eta_- - \frac{\mathrm{i}}{96}\left(\Omega_\eta{}^{mnp} H_{mnp}\right)\gamma_\mu\sigma^3\epsilon_- \otimes \eta_- \\
&\qquad\qquad - \frac{\mathrm{i}}{96}\left(\mathrm{e}^\phi\,\Omega_\eta{}^{mnp} F_{mnp}\right)\gamma_\mu\sigma^1\epsilon_- \otimes \eta_- \ ,
\end{aligned} \tag{2.122}$$

where the second equation used (2.12) and the decomposition of the ten-dimensional gamma-matrices given in (2.2). Similarly we compute

$$\begin{aligned}
\frac{1}{2}\Pi_-\,\Gamma_\mu{}^m\delta\hat{\Psi}_m &= -\left(\bar{\eta}_-\gamma^m D_m\eta_+\right)\gamma_\mu\varepsilon_- \otimes \eta_- + \frac{1}{16}\left(\bar{\eta}_-\gamma^{mnp}\eta_+ H_{mnp}\right)\gamma_\mu\sigma^3\varepsilon_- \otimes \eta_- \\
&\qquad\qquad + \frac{1}{16}\left(\mathrm{e}^\phi\,\bar{\eta}_-\gamma^{mnp}\eta_+ F_{mnp}\right)\gamma_\mu\sigma^1\varepsilon_- \otimes \eta_- \\
&= -\frac{3\mathrm{i}}{4}W_1\gamma_\mu\varepsilon_- \otimes \eta_- + \frac{\mathrm{i}}{32}\left(\Omega_\eta{}^{mnp} H_{mnp}\right)\gamma_\mu\sigma^3\varepsilon_- \otimes \eta_- \\
&\qquad\qquad + \frac{\mathrm{i}}{32}\left(\mathrm{e}^\phi\,\Omega_\eta{}^{mnp} F_{mnp}\right)\gamma_\mu\sigma^1\varepsilon_- \otimes \eta_- \ .
\end{aligned} \tag{2.123}$$

In the second equality we used $D_m\eta_+ = \frac{i}{4}W_1 g_{mn}\gamma^n\eta_- + \ldots$ (see [23]), which follows from (2.12) and (2.15). We omitted terms involving the other torsion classes that vanish when inserted in the bilinear expression $\bar{\eta}_- \gamma^m D_m \eta_+$. From (2.15) we read off W_1 to be

$$W_1 = -\frac{\mathrm{i}}{36} (\mathrm{d}J)_{mnp} \, \Omega_\eta{}^{mnp} \ . \tag{2.124}$$

Inserting (2.122) and (2.123) into (2.116) and comparing to (2.113) we arrive at

$$\begin{aligned} S_{11} &= \frac{1}{48} (\mathrm{i}\, \mathrm{d}J + H)_{mnp} \, \Omega_\eta{}^{mnp} \ , \\ S_{22} &= \frac{1}{48} (\mathrm{i}\, \mathrm{d}J - H)_{mnp} \, \Omega_\eta{}^{mnp} \ , \\ S_{12} = S^{21} &= \frac{1}{48} e^\phi \, F_{mnp} \, \Omega_\eta{}^{mnp} \ . \end{aligned} \tag{2.125}$$

We can write the matrix S in a compact form, which uses the Mukai pairing defined in (2.38):

$$S_{AB}(\mathrm{IIB}) = \frac{\mathrm{i}}{8} \begin{pmatrix} W + P & Q_\mathrm{B} \\ Q_\mathrm{B} & W - P \end{pmatrix} \tag{2.126}$$

where

$$\begin{aligned} W \epsilon_g &\equiv \left\langle \Omega_\eta, \mathrm{d}e^{\mathrm{i}J} \right\rangle \ , \\ P \epsilon_g &\equiv \left\langle \Omega_\eta, H_3 \right\rangle \ , \\ Q_\mathrm{B} \, \epsilon_g &\equiv \mathrm{e}^\phi \left\langle \Omega_\eta, F_\mathrm{B} \right\rangle \ . \end{aligned} \tag{2.127}$$

Here we have used

$$\epsilon_g = \frac{1}{6} J \wedge J \wedge J = \frac{\mathrm{i}}{8} \, \Omega_\eta \wedge \bar{\Omega}_\eta \tag{2.128}$$

denoting the volume form defined by the (string frame) metric g_{mn}, and $F_\mathrm{B} = F_1 + F_3 + F_5$ is the sum of all IIB RR field strengths defined in (2.120) (out of which only F_3 contributes to the superpotential).

A similar calculation can be done for type IIA, where we need to use both Π_+ and Π_- since the two ten-dimensional gravitinos have opposite chiralities. For the RR piece, we get the following terms in the supersymmetry transformation

$$\begin{aligned} \sum_{n \text{ even}} \frac{(-1)^{n/2}}{n!} \, \mathrm{e}^\phi F_{p_1 \ldots p_n} \bar{\eta}_+ \gamma^{p_1 \ldots p_n} \eta_+ &= \sum_{n \text{ even}} \frac{1}{n!} \, \mathrm{e}^\phi F_{p_1 \ldots p_n} \bar{\eta}_- \gamma^{p_1 \ldots p_n} \eta_- \\ &= \frac{1}{2} e^\phi \left(F_0 - \frac{\mathrm{i}}{2} F_{ab} J^{ab} - \frac{1}{8} F_{abcd} J^{ab} \, J^{cd} + \frac{\mathrm{i}}{48} F_{abcdef} J^{ab} \, J^{cd} \, J^{ef} \right) . \end{aligned} \tag{2.129}$$

This term can also be written using the Mukai pairing defined in (2.38) as[9]

$$\left(F_0 - \frac{\mathrm{i}}{2} F_{ab} J^{ab} - \frac{1}{8} F_{abcd} J^{ab} \, J^{cd} + \frac{\mathrm{i}}{48} F_{abcdef} J^{ab} \, J^{cd} \, J^{ef} \right) \epsilon_g = \left\langle \mathrm{e}^{-\mathrm{i}J}, F_\mathrm{A} \right\rangle . \tag{2.130}$$

where $F_\mathrm{A} = F_0 + F_2 + F_4 + F_6$.

[9]Note that $\left\langle \mathrm{e}^{-\mathrm{i}J}, F_\mathrm{A} \right\rangle = [F \wedge \mathrm{e}^{\mathrm{i}J}]_6$, where the subscript 6 indicates the top form component, and the change in the sign of the exponent accounts for the alternating sign in the definition of the pairing, eq. (2.38).

To see explicitly the complexification of $\mathrm{e}^{-\mathrm{i}J}$ into e^{-t}, where $t = B + \mathrm{i}J$ as in (2.82), one transforms the RR terms into another field basis defined by $C = \mathrm{e}^B A$. This transforms the field strength defined in (2.120) according to

$$\begin{aligned} F_{2n} &= \mathrm{d}C_{2n-1} - H \wedge C_{2n-3} \\ &= (\mathrm{e}^B G_{\mathrm{A}})_{2n} = G_{2n} + B \wedge G_{2n-2} + \cdots + B^n \wedge G_0 \ , \end{aligned} \tag{2.131}$$

where we defined $G_{2n} = \mathrm{d}A_{2n-1}$ and $G_{\mathrm{A}} = G_0 + G_2 + G_4 + G_6$. In this basis the expression (2.130) is replaced by $\langle \mathrm{e}^{-\mathrm{i}J}, F_{\mathrm{A}} \rangle = \langle \mathrm{e}^{-(B+\mathrm{i}J)}, G_{\mathrm{A}} \rangle$.

Collecting all the pieces together, we get for type IIA the following matrix S:

$$S_{AB}(\mathrm{IIA}) = \frac{\mathrm{i}}{8} \begin{pmatrix} \bar{W} - \bar{P} & -\bar{Q}_{\mathrm{A}} \\ -\bar{Q}_{\mathrm{A}} & W + P \end{pmatrix}, \tag{2.132}$$

where P and W are defined as in IIB ($\bar{P}$, $\bar{W}$ are their complex conjugates), and

$$\bar{Q}_{\mathrm{A}}\, \epsilon_g = \mathrm{e}^{\phi} \left\langle \mathrm{e}^{-(B+\mathrm{i}J)}, G_{\mathrm{A}} \right\rangle . \tag{2.133}$$

Using the constraints (2.105) $J \wedge \Omega_\eta = B \wedge \Omega_\eta = 0$ we can rewrite the W and P terms as

$$\begin{aligned} W \epsilon_g &\equiv \langle \mathrm{e}^{\mathrm{i}J}, \mathrm{d}\Omega_\eta \rangle \ , \\ P \epsilon_g &\equiv \langle \mathrm{e}^{B}, \mathrm{d}\Omega_\eta \rangle \ . \end{aligned} \tag{2.134}$$

To complete the analysis we would like to write the expressions for S_{AB} in terms of the Kähler potentials K_J and K_ρ and the unnormalized pure spinor fields $2c\,\mathrm{e}^{-B-\mathrm{i}J}$ and Ω. First we recall that the natural metric (2.21) on $T^{1,3}$ is $g^{(4)}_{\mu\nu} = \mathrm{e}^{-2\phi^{(4)}} g_{\mu\nu}$. Since there is a γ_μ term multiplying S_{AB} in the gravitino variation (2.113), the correctly normalized S_{AB} matrices should be multiplied by a factor of $\mathrm{e}^{\phi^{(4)}}$. Given the definitions (2.112) of K_J and K_ρ and the definition (2.21) of the four-dimensional dilaton $\phi^{(4)}$, we note that the different six-forms are related by

$$\epsilon_g = \frac{1}{8|c|^2}\mathrm{e}^{-K_J} = \frac{1}{8n^2}\mathrm{e}^{-K_\rho} = \mathrm{e}^{-2\phi^{(4)}+2\phi} \ . \tag{2.135}$$

Thus the correctly normalized IIA and IIB matrices S can be written as

$$S^{(4)}_{AB}(\mathrm{IIA}) = \mathrm{i}\,\mathrm{e}^{\frac{1}{2}K_J} \begin{pmatrix} -\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}} \langle \Phi^+, \mathrm{d}\bar{\Phi}^- \rangle & -\dfrac{1}{2\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \langle \Phi^+, G_{\mathrm{A}} \rangle \\ -\dfrac{1}{2\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \langle \Phi^+, G_{\mathrm{A}} \rangle & \mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}} \langle \Phi^+, \mathrm{d}\Phi^- \rangle \end{pmatrix}, \tag{2.136}$$

$$S^{(4)}_{AB}(\mathrm{IIB}) = \mathrm{i}\,\mathrm{e}^{\frac{1}{2}K_\rho} \begin{pmatrix} -\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}} \langle \Phi^-, \mathrm{d}\Phi^+ \rangle & \dfrac{1}{2\sqrt{2}}\mathrm{e}^{2\phi^{(4)}} \langle \Phi^-, G_{\mathrm{B}} \rangle \\ \dfrac{1}{2\sqrt{2}}\mathrm{e}^{2\phi^{(4)}} \langle \Phi^-, G_{\mathrm{B}} \rangle & \mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}} \langle \Phi^-, \mathrm{d}\bar{\Phi}^+ \rangle \end{pmatrix}, \tag{2.137}$$

where

$$\Phi^+ = c\,\mathrm{e}^{-B-\mathrm{i}J}, \qquad \Phi^- = \Omega. \tag{2.138}$$

In deriving the expressions (2.136) and (2.137) we have used the fact that the constraints $B\wedge\Omega = J\wedge\Omega = 0$ imply that there are no contributions from terms with $\mathrm{d}c$ or $\mathrm{d}\bar{c}$. We have also made a $U(1)_R \otimes \mathrm{SU}(2)_R$ R-symmetry transformation to remove explicit dependence on the phase of c. These constraints also ensure the invariance of $S^{(4)}_{AB}$ under rescalings of c and Ω. Finally, we have also used the fact that $B\wedge\Omega = 0$ to replace F_{B} with G_{B}, defined in exact analogy to G_{A} in (2.131).[10] Recalling that e^{-K_J}, e^{-K_ρ} and $\mathrm{e}^{-2\phi^{(4)}}$ are all six-forms, overall, $S^{(4)}_{AB}$ transforms as a section of $(\Lambda^6 F^*)^{-1/2}$. This dependence arises because of the rescaling by $\mathrm{e}^{\phi^{(4)}}$.

Comparing (2.136) and (2.137) with (2.114), and recalling that for IIA $K_V = K_J$ while for IIB $K_V = K_\rho$, we can read off the Killing prepotentials $\mathcal{P}^x$. For type IIA we obtain

$$\begin{aligned} \mathcal{P}^1 &= -2\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}}\left\langle \Phi^+, \mathrm{d}\,\mathrm{Re}\,\Phi^-\right\rangle ,\\ \mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}}\left\langle \Phi^+, \mathrm{d}\,\mathrm{Im}\,\Phi^-\right\rangle ,\\ \mathcal{P}^3 &= \frac{1}{\sqrt{2}}\mathrm{e}^{2\phi^{(4)}}\left\langle \Phi^+, G_{\mathrm{A}}\right\rangle , \end{aligned} \tag{2.139}$$

while for type IIB we find

$$\begin{aligned} \mathcal{P}^1 &= -2\,\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}}\left\langle \Phi^-, \mathrm{d}\,\mathrm{Re}\,\Phi^+\right\rangle ,\\ \mathcal{P}^2 &= 2\,\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}}\left\langle \Phi^-, \mathrm{d}\,\mathrm{Im}\,\Phi^+\right\rangle ,\\ \mathcal{P}^3 &= -\frac{1}{\sqrt{2}}\mathrm{e}^{2\phi^{(4)}}\left\langle \Phi^-, G_{\mathrm{B}}\right\rangle . \end{aligned} \tag{2.140}$$

Note that in both cases that $\mathcal{P}^x$ transform as scalars since though the Mukai pairing is an element of $\Lambda^6 F^*$, this dependence is canceled by the Kähler potential and $\phi^{(4)}$ factors.

If we turn off the RR fluxes we see that in both cases only $\mathcal{P}^1$ and $\mathcal{P}^2$ are non-zero. Since the $\mathcal{P}^x$ are SU(3) singlets only singlet terms can contribute. In particular, from (2.15) and (2.16) we see they only depend on the singlet torsion class $W_1 \sim \mathrm{d}J\wedge\Omega$ as we also already observed in (2.124). Furthermore W_1 is complexified by $H\wedge\Omega$, the singlet part of the NS flux. The RR fluxes, which in both cases enter $\mathcal{P}^3$, appear only in the singlet representation.

Eqs. (2.139) and (2.140) are quite generic. The only restriction was that we assumed there is a single SU(3) structure and not the more general situation of an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure. The key difference between these cases is that Φ^- is not generic but contains only the holomorphic three-form piece Ω. More generally it would include one- and five-form pieces. The precise details of the corresponding derivation of the prepotentials $\mathcal{P}^x$ are beyond the scope of this paper and will be presented elsewhere. Nonetheless, given the simple form the final expressions in terms of $\Phi^\pm$ we conjecture that (2.139) and (2.140) actually continue to hold also when $\Phi^\pm$ are generic $\mathrm{SU}(3)\times\mathrm{SU}(3)$ pure spinors.

Let us also briefly discuss mirror symmetry which states that type IIA and type IIB are equivalent when considered in mirror symmetric backgrounds. For a pair of mirror

[10] Later in section 3 we will split the field strengths into an exact piece, plus a flux piece (see, for example, (3.32)).

Calabi-Yau manifolds even and odd cohomologies are interchanged $H^{\rm even} \leftrightarrow H^{\rm odd}$. It has been suggested that on manifolds with $SU(3)$ structure this operation is replaced by an exchange of the two pure spinors $\Omega \leftrightarrow e^{-B-iJ}$ and the exchange of $\Lambda^{\rm even}F^* \leftrightarrow \Lambda^{\rm odd}F^*$ [23]. More generally we expect the exchange of SU(3) × SU(3) pure spinors and fluxes

$$\Phi^+ \leftrightarrow \Phi^-, \qquad G_{\rm A} \leftrightarrow G_{\rm B}. \tag{2.141}$$

In the generic SU(3) × SU(3) case, we see that (2.139) and (2.140) indeed have this symmetry (provided we also exchange K_J and K_ρ).[11] However, recall that the expressions were derived assuming only an SU(3)-structure where Φ^- is restricted to be the three-form Ω while $\Phi^+ = c\, e^{-B-iJ}$ is generic. In this case Ω and e^{-B-iJ} enter asymmetrically into (2.139) and (2.140). As we already observed in $\mathcal{P}^1$ and $\mathcal{P}^2$ only $d(B+iJ)\wedge\Omega$ appears. Thus the three-form Ω fully contributes while out of e^{-B-iJ} only the two-form $B+iJ$ but not its square $(B+iJ)^2$ appears. This is precisely the issue of the missing mirror of the magnetic fluxes. In [13, 15] it was observed that $d\Omega$ can be interpreted as a NS four-form which is the mirror dual of the NS three-form H. However in order to have full mirror symmetry a NS two-form is also necessary. The four-form can be viewed as giving rise to electric fluxes while the (missing) two-form would lead to magnetic fluxes. Within the framework followed here the NS two-form and thus the magnetic fluxes are still missing. From the structure of (2.139) and (2.140) it is clear the NS two-form should be identified with the exterior derivative of the one-form part of Φ^- which exists in backgrounds with generic SU(3) × SU(3) structure.[12] This is also in accord with similar recent suggestions for example in ref. [81, 82]. We will return to this issue again in section 3.6.

2.6 $N = 1$ superpotentials

In the spirit followed so far in this section we can further constrain the space-time background to only realize four supercharges linearly. This results in a further split of the $N = 2$ multiplets which we discussed in section 2.2. In particular, the $N = 2$ gravitational multiplet decomposes into a $N = 1$ gravitational multiplet containing the metric and one gravitino $(g_{\mu\nu}, \psi_\mu)$, and a $N = 1$ spin-$\frac{3}{2}$ multiplet containing the second gravitino and the graviphoton (ψ'_μ, C_μ). Exactly as in $N = 2$, the appearance of a standard $N = 1$-type action requires that we project out the $N = 1$ spin-$\frac{3}{2}$ multiplet leaving only the gravitational multiplet together with vector, tensor and chiral multiplets in the spectrum.[13] From a supergravity point of view such a truncation has been discussed in refs. [84] while the truncation occurring in orientifold compactifications of type II has been studied in ref. [85, 86].

We compute the $N = 1$ superpotential $\mathcal{W}$ from the supersymmetry transformation of the linear combination of the two $N = 2$ gravitinos which resides in the $N = 1$ gravitational multiplet. Exactly which linear combination is kept depends on the specific theory under

[11] For the expressions given we also have to map $(\mathcal{P}^2, \mathcal{P}^3) \to (-\mathcal{P}^2, -\mathcal{P}^3)$ in going from IIA to IIB, but this is just an element of the $SU(2)_R$ symmetry and is simply an artifact of our conventions.

[12] This has also been noticed by T. Grimm and we thank him for discussions on this point.

[13] Alternatively one can integrate out the spin-$\frac{3}{2}$ multiplet and consider an effective action below the scale set by the mass of that multiplet [83].

consideration. As a consequence $\mathcal{W}$ will depend on two angles which parameterize the choices of embedding an $N=1$ inside $N=2$.

We proceed by choosing the $N=1$ supersymmetry transformation parameter ε to be any linear combination of the pair of $N=2$ parameters ε^1 and ε^2. We parameterize this freedom by writing

$$\varepsilon_A = \varepsilon n_A, \qquad n_A = \begin{pmatrix} a \\ b \end{pmatrix} \ , \quad |a|^2 + |b|^2 = 1 \ , \tag{2.142}$$

where ε is the $N=1$ supersymmetry parameter and n_A is a vector normalized to one so $\bar{n}^A n_A = 1$ with a and b complex. Choosing such a linear combination breaks the $\mathrm{SU}(2)_R$ symmetry of $N=2$ down to a $U(1)_R$ of $N=1$, corresponding to those rotations preserving n^A. The conjugated spinors ε^A can be written as $\varepsilon^A = \varepsilon^c n^{*A}$ where ε^c is the conjugate $N=1$ spinor, and $n^{*A} = \binom{\bar{a}}{\bar{b}}$. We can similarly decompose the gravitinos $\psi_{A\mu}$. If ψ_μ is the $N=1$ gravitino, the superpotential $\mathcal{W}$ can be extracted from the supersymmetry variation which has the generic form

$$\delta\psi_\mu = D_\mu\varepsilon + \mathrm{i}\mathrm{e}^{K/2}\mathcal{W}\gamma_\mu\varepsilon^c \ , \tag{2.143}$$

where K is the total $N=1$ Kähler potential. We can use the projector $\Pi_A{}^B = n_A\bar{n}^B$ to pick out the $N=1$ gravitino and supersymmetry parameter. Projecting the $N=2$ variation (2.113) we find

$$\delta\psi_\mu = D_\mu\varepsilon + \mathrm{i}\bar{n}^A S_{AB} n^{*B}\gamma_\mu\varepsilon^c \ , \tag{2.144}$$

Comparing (2.144) with (2.143) using (2.114) we arrive at

$$\begin{aligned} \mathrm{e}^{K/2}\mathcal{W} = \frac{\mathrm{i}}{2}\mathrm{e}^{K_V/2}\Big[& \left(\cos^2\alpha\,\mathrm{e}^{\mathrm{i}\beta} - \sin^2\alpha\,\mathrm{e}^{-\mathrm{i}\beta}\right)\mathcal{P}^1 \\ & - \mathrm{i}\left(\cos^2\alpha\,\mathrm{e}^{\mathrm{i}\beta} + \sin^2\alpha\,\mathrm{e}^{-\mathrm{i}\beta}\right)\mathcal{P}^2 - \sin 2\alpha\mathcal{P}^3\Big], \end{aligned} \tag{2.145}$$

To get this expression, we have used the fact that the superpotential depends on the sum of the arguments of a and b only by an overall phase, which can be removed by a $U(1)_R$ transformation. We can therefore parameterize a and b using only two angles, α, β as

$$a = \cos\alpha\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\beta} \ , \qquad b = \sin\alpha\,\mathrm{e}^{\frac{\mathrm{i}}{2}\beta} \ . \tag{2.146}$$

From (2.145) we see that in both cases, type IIA and type IIB, the $N=1$ superpotential depends on the fluxes and the torsion class W_1 via $\mathcal{P}^x$. In addition it also depends on the two angles α, β which fix a $U(1)_R$ subgroup inside the $SU(2)_R$ of $N=2$. The fact that a classification of backgrounds with SU(3) structure in type II needs only two angles was indeed anticipated in [26].

In order to give the $N=1$ Kähler potential in terms of chiral multiplets one first needs to determine the complex structure on the field space which in general is an involved procedure [62, 85, 86]. However, here we do not need K in any explicit form but instead it is sufficient to give it in terms of $N=2$ quantities. By inspecting the appropriately normalized (i.e. Weyl rescaled) gravitino mass term one determines the generic relation

$$K = K_J + K_\rho + 2\phi^{(4)} \ . \tag{2.147}$$

Let us stress once more that (2.147) does not express K in proper $N = 1$ chiral coordinates but in terms of $N = 2$ coordinates. However this is all we need in order to insert (2.147) into (2.145) and using (2.139), (2.140) we arrive at

$$\mathrm{i}\mathcal{W}_{\mathrm{IIA}} \;=\; \cos^2\alpha\,\mathrm{e}^{\mathrm{i}\beta}\langle\Phi^+,\mathrm{d}\bar{\Phi}^-\rangle \,-\, \sin^2\alpha\,\mathrm{e}^{-\mathrm{i}\beta}\langle\Phi^+,\mathrm{d}\Phi^-\rangle \,+\, |n|\sin 2\alpha\,\mathrm{e}^{\phi}\left\langle\Phi^+,G_{\mathrm{A}}\right\rangle, \quad (2.148)$$

and

$$\mathrm{i}\mathcal{W}_{\mathrm{IIB}} \;=\; \cos^2\alpha\,\mathrm{e}^{\mathrm{i}\beta}\left\langle\Phi^-,\mathrm{d}\Phi^+\right\rangle - \sin^2\alpha\,\mathrm{e}^{-\mathrm{i}\beta}\left\langle\Phi^-,\mathrm{d}\bar{\Phi}^+\right\rangle - |c|\sin 2\alpha\,\mathrm{e}^{\phi}\left\langle\Phi^-,G_{\mathrm{B}}\right\rangle. \quad (2.149)$$

Note that we have written these expressions in terms of the ten-dimensional dilaton ϕ.

For specific choices of α, β we can reproduce the $N = 1$ superpotentials discussed so far in the literature. If we choose $2\alpha = -\beta = \pi/2$ and undo the change of variables from G_3 to F_3 in (2.149) we obtain the Gukov-Taylor-Vafa-Witten superpotential [70, 59][14]

$$\mathcal{W}_{\mathrm{GTVW}} = \mathrm{i}\,|c|\,\mathrm{e}^{\phi}\,\langle(F_3 - \tau H_3),\Omega\rangle\;, \quad (2.150)$$

where $\tau = C_0 + \mathrm{i}\mathrm{e}^{-\phi}$.

On the type IIA side the mirror symmetric superpotential for only RR-fluxes as suggested in [71] is obtained from (2.148) for $\alpha = \pi/4$ and $\mathrm{d}\Phi^- = 0$

$$\mathcal{W}_{\mathrm{IIA,RR}} = -\mathrm{i}\,|c|n\,\mathrm{e}^{\phi}\left\langle\mathrm{e}^{-(B+\mathrm{i}J)},G_{\mathrm{A}}\right\rangle\;. \quad (2.151)$$

The type IIA mirror superpotential of the NS-fluxes was proposed in [15] and it can be recovered from (2.148) with the choice $\alpha = \pi/2$ and $\beta = -\pi/2$

$$\mathcal{W}_{\text{half-flat}} = |c|\left\langle\mathrm{e}^{-(B+\mathrm{i}J)},\mathrm{d}\Omega\right\rangle\;. \quad (2.152)$$

Finally, the superpotential proposed in ref. [42] is expressed in terms of the periods of a mirror pair of Calabi-Yau threefolds. The resulting structure is similar to the superpotentials (2.148), (2.149) and it would be interesting to establish a precise relationship.

Having obtained the most general $N = 1$ superpotential, let us go back to $N = 2$ and discuss the truncation to four space-time dimensions.

3. The $N = 2$ effective theory in four dimensions

In the previous sections we studied the ten-dimensional type II supergravities in spacetimes $M^{1,9}$ where it is possible to single out eight of the original 32 supercharges. However so far we simply rewrote the ten-dimensional theory keeping all of the modes in the theory. In other words we did not yet perform any Kaluza–Klein reduction. In fact, we have not even assumed that $M^{1,9}$ is a product. Generically it only required that $M^{1,9}$ admitted an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure, or, in the special case we considered in detail, simply an SU(3)-structure.

[14] The NS contribution to this superpotential was verified in the heterotic theory using the supersymmetry transformation of the gravitino by a calculation similar to the one we do to obtain the $N = 2$ prepotential in [65]. This "GVW choice" corresponds to the relation $a = -ib$, which gives precisely the $N = 1$ conserved spinor in compactifications on (conformal) CY with fluxes [87].

In this section we turn to a more restricted situation. First we will assume that we have topologically a product manifold

$$M^{1,9} = M^{1,3} \times Y \ . \tag{3.1}$$

Comparing with the generic case (2.6), we identify

$$T^{1,3} = TM^{1,3} \ , \qquad F = TY \ . \tag{3.2}$$

We then truncate the general ten-dimensional eight-supercharge theory, keeping only a finite number of "light" modes in the spectrum. This in turn will lead to a four-dimensional effective theory with $N = 2$ supersymmetry. For simplicity, we will not discuss the general situation where there is an $SU(3) \times SU(3)$ structure but instead confine our attention to backgrounds Y with only $SU(3)$ structure. In addition we define the truncation in such a way that apart from the gravitational multiplet only vector-, tensor- and hypermultiplets are present in the effective theory. In particular we project out all possible spin-$\frac{3}{2}$ multiplets and in this way end up with a "standard" $N = 2$ effective action.

3.1 Defining the truncation

Generically the distinction between heavy and light modes in a Kaluza–Klein expansion on the product (3.1) is not straightforward. This is in contrast to the situation when Y is a Calabi-Yau manifold. In this case one keeps all the field deformations which from a four-dimensional point of view are massless modes. For each supergravity field these correspond to harmonic forms, and hence the light (massless) modes are finite in number. For instance, of the metric deformations given in section 2.3.2, the massless modes are in one-to-one correspondence with harmonic deformations of the Kähler-form J and the three-form ρ. These give the $h^{1,1}$ Kähler moduli and $h^{2,1}$ complex structure moduli, where $h^{1,1}, h^{2,1}$ are the appropriate Hodge numbers. Similarly massless deformation of B_{mn} and the RR-potentials C_p are in one-to-one correspondence with harmonic two- and p-forms respectively. The result is that, for a Calabi-Yau compactification, rather than considering J, ρ, B and C_p to be forms on Y we truncate to the finite dimensional sub-space of harmonic forms.

This suggests that we should take a similar truncation in the generic case – not to harmonic forms but some other finite-dimensional subspace of $\Lambda^* F^*$. Identifying the subspace however is not a simple task. If we start with a background (like a Calabi-Yau manifold) which satisfies the equations of motion, we can truncate to the massless fluctuations. However generically the background is not a solution. An example of this is a Calabi-Yau compactification with non-zero H or RR flux. Typically in this case one still keeps the harmonic deformations even though some of them become massive. This can be justified in the limit of large manifolds and small fluxes by the fact that there is a hierarchy between these masses and those of the Kaluza–Klein modes. Similar arguments can be made for mirror half-flat manifolds in the large complex structure limit [15]. Here, for now, we will simply assume that a suitable limit can be found where it is consistent to keep only a finite number of "light" modes and not specify how this subset is defined.

Let us be more specific. We wish to restrict to a set of finite-dimensional subspaces of $\Lambda^p F^*$. We write these as

$$\Lambda^p_{\text{finite}} \subset \Lambda^p F^* \tag{3.3}$$

and assume that all the fields g, B, ϕ and C_p take values in these subspaces. In particular, to describe the metric degrees of freedom, we assume that we have $O(6,6)$ spinors $\chi^+ = 2\,\mathrm{Re}(c\,\mathrm{e}^{-B-\mathrm{i}J}) \in \Lambda^{\text{even}}_{\text{finite}}$ and $\chi^- = 2n\rho \in \Lambda^3_{\text{finite}}$. We then define the spaces of stable forms

$$\begin{aligned} U^{\text{finite}}_J &= U_J \cap \Lambda^{\text{even}}_{\text{finite}}, \\ U^{\text{finite}}_\rho &= U_\rho \cap \Lambda^3_{\text{finite}} \end{aligned} \tag{3.4}$$

where U_J and U_ρ are the spaces of stable forms defined in (2.77) and (2.88). We assume that U^{finite}_J and U^{finite}_ρ are open subsets of $\Lambda^{\text{even}}_{\text{finite}}$ and $\Lambda^3_{\text{finite}}$, which is generically the case.

Crucially the truncation should not break supersymmetry. This means that the special Kähler metrics on the spaces U_J and U_ρ give special Kähler metrics on U^{finite}_J and U^{finite}_ρ. This is equivalent to requiring

1. the Mukai pairing $\langle \cdot, \cdot \rangle$ is non-degenerate on $\Lambda^{\text{even/odd}}_{\text{finite}}$,
2. if $\chi^\pm \in U^{\text{finite}}_J$ or U^{finite}_ρ then $\hat{\chi}^\pm \in U^{\text{finite}}_J$ or U^{finite}_ρ,

where the Mukai pairing is defined in (2.38) and $\hat{\chi}^\pm$ are defined in (2.52) and (2.90). The first condition implies we have symplectic structures on U^{finite}_J and U^{finite}_ρ, the second that we have complex structures. Note that the second condition is equivalently to

2.′ if $\chi^\pm \in U^{\text{finite}}_J$ or U^{finite}_ρ then $*\chi^\pm \in U^{\text{finite}}_J$ or U^{finite}_ρ,

where $*$ is the Hodge-star operator defined by the metric g (which is in turn defined by the particular χ^+ and χ^-).

As in the Calabi-Yau case we can define $\Lambda^p_{\text{finite}}$ in terms of sets of basis forms. For instance for $\Lambda^{2p}_{\text{finite}} \subset \Lambda^{2p} F^*$ we write

$$\begin{aligned} \Lambda^0 F^* &\supset \Lambda^0_{\text{finite}} = \big\{\text{constant functions on } Y\big\} \\ \Lambda^2 F^* &\supset \Lambda^2_{\text{finite}} = \{A^a \omega_a,\ a = 1,\dots,b_J\} \\ \Lambda^4 F^* &\supset \Lambda^4_{\text{finite}} = \{B_a \tilde{\omega}^a,\ a = 1,\dots,b_J\} \\ \Lambda^6 F^* &\supset \Lambda^6_{\text{finite}} = \{C\epsilon\} \end{aligned} \tag{3.5}$$

where ω_a are a set of basis two-forms, $\tilde{\omega}^a$ a set of basis four-forms, ϵ is a volume form and A^a, B_a and C are constant functions on Y. The condition that the Mukai pairing is non-degenerate on $\Lambda^{\text{even}}_{\text{finite}}$ is reflected in the fact that $\Lambda^2_{\text{finite}}$ and $\Lambda^4_{\text{finite}}$ have the same dimension b_J. Specifically we choose the basis $(1, \omega_a, \tilde{\omega}^b, \epsilon)$ such that

$$\left\langle \omega_a, \tilde{\omega}^b \right\rangle = -\delta_a{}^b \epsilon\ , \qquad a, b = 1, \dots, b_J\ , \tag{3.6}$$

(Recall all other products vanish identically, except for $\langle 1, \epsilon \rangle$ which equals ϵ by definition.) Note that this expansion is not quite general in that we have assumed that $\Lambda^0_{\text{finite}}$ is spanned by constant functions on Y. The reason for this will be explained below. Finally, in order

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to impose the second condition 2.′ above, in analogy with the Calabi-Yau case where ω_a and $\tilde{\omega}^a$ are harmonic, we allow the basis vectors in general to depend on the metric g_{mn} so that

$$*1 \in \Lambda^6_{\text{finite}} , \qquad *\omega_a \in \Lambda^4_{\text{finite}} , \qquad *\tilde{\omega}^a \in \Lambda^2_{\text{finite}} , \tag{3.7}$$

for all a.

For the odd forms we choose a more restricted truncation. First for the three-forms we define, as above,

$$\Lambda^3 F^* \supset \Lambda^3_{\text{finite}} = \left\{ D^K \alpha_K + E_L \beta^L,\ K, L = 0, \dots, b_\rho \right\} \tag{3.8}$$

where D^K and E_L are constant functions on Y and $\alpha_K, \beta^L \in \Lambda^3 F^*$ are a symplectic set of basis forms satisfying

$$\left\langle \alpha_K, \beta^L \right\rangle = \delta_K{}^L \epsilon , \qquad K, L = 1, \dots, b_\rho , \tag{3.9}$$

while $\langle \alpha_K, \alpha_L \rangle = \left\langle \beta^K, \beta^L \right\rangle = 0$. In addition we require

$$*\alpha_K, *\beta^L \in \Lambda^3_{\text{finite}} \tag{3.10}$$

for all K and L.

At this point we further simplify the truncation by imposing an additional condition. For the one- and five-forms we choose to truncate the spectrum completely, keeping no light modes, so

$$\Lambda^1_{\text{finite}} = \Lambda^5_{\text{finite}} = 0. \tag{3.11}$$

We make this choice because we want to truncate in such a way that a 'standard' $N = 2$ gauged supergravity appears which only contains one gravitational multiplet together with vector, tensor and hypermultiplets. In particular we do not allow the presence of any spin-$\frac{3}{2}$ multiplets. In terms of the gravitinos this amounts to keeping only the two gravitinos in the gravitational multiplet but projecting out all other gravitinos which may reside in spin-$\frac{3}{2}$ multiplets.

From the decomposition of the ten-dimensional spinors given in (2.11) and tables 1 and 4 we see that the SU(3) singlets correspond to the gravitinos in the $N = 2$ gravitational multiplet while the SU(3) triplets lead to gravitinos which reside in their own spin-$\frac{3}{2}$ multiplets. Of course the triplets are nothing but $(1, 0)$-forms on Y with respect to the given complex structure. Therefore excluding spin-$\frac{3}{2}$ multiplets in the truncation we are led to project out all modes arising from one-forms (or triplets) on Y. In this case one is left with the multiplets given in tables 5 and 6. For consistency it also implies that we should not be able to construct any **3**-representations from the bases (α_K, β^L) and $(1, \omega_a, \tilde{\omega}^a, \epsilon)$. This means that any five-form wedge products must vanish, so

$$\omega_a \wedge \alpha_K \ = \ 0 \ = \ \omega_a \wedge \beta^K , \qquad \forall\, a, K . \tag{3.12}$$

Thus $J \wedge \rho = 0$ holds identically for all $J \in \Lambda^2_{\text{finite}}$ and $\rho \in \Lambda^3_{\text{finite}}$.[15]

[15] This also removes the subtleties involved in the fact that generically multiple SU(3)-structures (J, ρ) on Y determine the same metrics g_{mn}: here, up to an overall rescaling of $\Omega_\eta = \rho + \mathrm{i}\hat{\rho}$ by $\mathrm{e}^{\mathrm{i}\alpha}$, each metric corresponds to a unique pair $(J, \rho) \in \Lambda^2_{\text{finite}} \oplus \Lambda^3_{\text{finite}}$.

Our assumption is that, in the truncation, all fields will be expanded in terms of elements of $\Lambda^*_{\rm finite}$. As we have seen in section 2.5, the supergravity action also depends on the intrinsic torsion $\mathrm{d}\Omega$ and $\mathrm{d}J$ through the superpotential terms. Similarly the field strengths H and F_p are written in terms of exterior derivatives. In order for the truncation to make sense all such terms also need to be in the truncated set $\Lambda^*_{\rm finite}$. In other words we require $\Lambda^*_{\rm finite}$ to be closed under d, that is

$$\mathrm{d} : \Lambda^p_{\rm finite} \to \Lambda^{p+1}_{\rm finite} \ . \tag{3.13}$$

Since $\Lambda^1_{\rm finite} = 0$ this means that $\Lambda^0_{\rm finite}$ must contain only constant functions, as, in fact, we have already assumed. Since $\Lambda^5_{\rm finite} = 0$, the condition (3.13) implies that the rest of the forms in the basis of $\Lambda^{\rm even}_{\rm finite}$ satisfy

$$\begin{aligned} \mathrm{d}\omega_a &= m_a^K\, \alpha_K + e_{aL}\, \beta^L \ , \\ \mathrm{d}\tilde{\omega}^a &= 0 \ , \end{aligned} \tag{3.14}$$

where m_a^K and e_{aL} are constant matrices. Since the basis defined by (3.9) is only specified up to symplectic transformations the matrices m_a^K and e_{aL} also carry a representation of the symplectic group $\mathrm{Sp}(2b_\rho + 2)$ and naturally combine into the symplectic vectors $V_a := (e_{aK}, m_a^K)$.

Similarly expanding $\mathrm{d}\alpha_K$ and $\mathrm{d}\beta^K$, and using (3.12), have

$$\langle \omega_a, \mathrm{d}\alpha_K \rangle = -\omega_a \wedge \mathrm{d}\alpha_K = \mathrm{d}\omega_a \wedge \alpha_K = -\langle \alpha_K, \mathrm{d}\omega_a \rangle \ . \tag{3.15}$$

Together with (3.14) and a similar expression with $\mathrm{d}\beta^K$, this implies

$$\begin{aligned} \mathrm{d}\alpha_K &= e_{aK} \tilde{\omega}^a \ , \\ \mathrm{d}\beta^K &= -m_a^K \tilde{\omega}^a \ . \end{aligned} \tag{3.16}$$

Using $\mathrm{d}^2 = 0$ yields the consistency condition

$$m_a^K\, e_{bK} - e_{aK} m_b^K = V_a \cdot V_b = 0 \ , \tag{3.17}$$

or, in other words, the symplectic vectors V_a have to be null with respect to the symplectic inner product. The conditions (3.14), (3.16) and (3.17) have also been obtained in ref. [72] using consistency considerations of $N = 2$ gauged supergravity. We are also going to see their necessity from the requirement of four-dimensional gauge invariance in sections 3.3 and 3.4.

Finally, let us note that when Y is compact we have a natural map from $\Lambda^6_{\rm finite}$ to $\mathbb{R}$ given simply by integrating the six-form over Y. In particular, the Mukai pairing leads to a natural symplectic structure on $\Lambda^{\rm even/odd}_{\rm finite}$ given by

$$\omega(\psi^\pm, \chi^\pm) = \int_Y \langle \psi^\pm, \chi^\pm \rangle \, . \tag{3.18}$$

If we fix the volume form ϵ such that $\int_Y \epsilon = 1$ this coincides with the symplectic inner product $\omega(\psi^\pm_\epsilon, \chi^\pm_\epsilon)$ on the corresponding $\mathrm{Spin}(6,6)$ spinors. This is the symplectic structure which naturally appears in a Calabi-Yau truncation. Note that one then also has a natural definition of the Hitchin scalar functional

$$H_Y(\chi^\pm) = \int_Y H(\chi^\pm), \tag{3.19}$$

for forms $\chi^\pm \in \Lambda^{\text{even/odd}}$.

3.2 Reducing the Neveu-Schwarz sector

Let us first discuss the truncation of the Neveu-Schwarz sector since it is common to both type II theories. This sector contains the metric, the two-form B and the dilaton ϕ. Since we defined the truncation in such a way that all triplets are projected out we see from table 1 that only $g_{\mu\nu}$, g_{mn}, $B_{\mu\nu}$, B_{mn}, and ϕ survive in the NS-sector. The Spin(1, 3) singlets $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ trivially descend to the four-dimensional theory. In complete analogy to the generic relations (2.21), we define four-dimensional Einstein frame metric $g^{(4)}_{\mu\nu}$ and the dilaton $\phi^{(4)}$ which together with $B_{\mu\nu}$ becomes a member of a tensor multiplet in both type II theories as seen from tables 5 and 6. The difference now is that these fields depend only on the four space-time coordinates of $M^{1,3}$.

We already argued that instead of g_{mn} we can discuss the theory more conveniently in terms of J and ρ. Our definition of the truncation assumed that J (and B_{mn}) and ρ have an expansion in terms of a basis of $\Lambda^2_{\text{finite}}$ and $\Lambda^3_{\text{finite}}$ respectively. The conditions on the spaces $\Lambda^p_{\text{finite}}$ arose because we wanted the special Kähler metrics on U_J and U_ρ to descend to U_J^{finite} and U_ρ^{finite}. As a consequence we can take the results of section 2.4 to give expressions for the corresponding Kähler potentials.

Consider first the metric on U_J^{finite}. In the truncated space the complex $O(6,6)$ spinor Φ^+ has the expansion

$$\Phi^+ = c\,\mathrm{e}^{-B-\mathrm{i}J} = X^0 + X^a\omega_a - \mathcal{F}_a\tilde{\omega}^a - \mathcal{F}_0\epsilon \tag{3.20}$$

where if we expand

$$B + \mathrm{i}J = t^a\omega_a \tag{3.21}$$

then we have complex coordinates on U_J^{finite}

$$X^A = (X^0, X^a) = (c, -ct^a). \tag{3.22}$$

The Hitchin functional $H_Y(\Phi^+)$ is given by

$$\begin{aligned} H_Y &= \mathrm{i}\left(\bar{X}^A\mathcal{F}_A - X^A\bar{\mathcal{F}}_A\right) \\ &= \frac{4}{3}|c|^2\int_Y J\wedge J\wedge J = \frac{1}{6}\mathrm{i}|X^0|^2\kappa_{abc}(t-\bar{t})^a(t-\bar{t})^b(t-\bar{t})^c\ , \end{aligned} \tag{3.23}$$

where $\kappa_{abc} \equiv \int_Y \omega_a\wedge\omega_b\wedge\omega_c$ and $\mathcal{F}_A = (\mathcal{F}_0, \mathcal{F}_a)$.

For U_ρ^{finite}, the complex three-form Φ^- has the expansion

$$\Phi^- = \Omega = Z^K\,\alpha_K - F_L\,\beta^L\ , \tag{3.24}$$

now defining complex coordinates Z^K on U_ρ^{finite}. The corresponding Hitchin functional is

$$H_Y \ = \ \mathrm{i}\left(\bar{Z}^K F_K - Z^K\bar{F}_K\right) \ = \ -\mathrm{i}\int_Y \Omega\wedge\bar{\Omega}\ . \tag{3.25}$$

Since we have truncated in such a way to remove the triplet degrees of freedom, we necessarily satisfy the SU(3) conditions $J \wedge \rho = B \wedge \rho = 0$, as a result of eqn. (3.12). Furthermore, the extra deformations which modify the SU(3) structure but leave g invariant are also projected out. Thus none of the subtleties discussed in section 2.4.6 is of any concern here. Instead the moduli space of metric deformations is simply

$$\mathcal{M}^{\text{finite}} = \mathcal{M}_J^{\text{finite}} \times \mathcal{M}_\rho^{\text{finite}} \tag{3.26}$$

where $\mathcal{M}_J^{\text{finite}}$ and $\mathcal{M}_\rho^{\text{finite}}$ are the spaces U_J^{finite} and U_ρ^{finite} modulo rescalings of c in Φ^+ and the magnitude and phase of Ω in Φ^-, that is

$$\mathcal{M}_J^{\text{finite}} = U_J^{\text{finite}}/\mathbb{C}^* , \qquad \mathcal{M}_\rho^{\text{finite}} = U_\rho^{\text{finite}}/\mathbb{C}^* . \tag{3.27}$$

These spaces can be parametrized by the local "special" coordinates $t^a = X^a/X^0$ and $z^k = Z^K/Z^0$. In the latter case we isolate one (labeled α_0) of the α_K, and assume that we can consistently scale its coefficient to unity when expanding Ω. The corresponding Kähler potentials are

$$\begin{aligned} K_J &= -\ln H_Y = -\ln \int_Y \frac{4}{3}|c|^2 J \wedge J \wedge J = -\ln \mathrm{i}\left(\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A\right) , \\ K_\rho &= -\ln H_Y = -\ln \mathrm{i} \int_Y \Omega \wedge \bar{\Omega} = -\ln \mathrm{i}\left(\bar{Z}^K F_K - Z^K \bar{F}_K\right) . \end{aligned} \tag{3.28}$$

Note the natural scalar Hitchin functionals (3.19), allow us to define scalar Kähler potentials, as should appear in a four-dimensional theory, rather than the six-forms e^{-K_J} and e^{-K_ρ} that appear in the general ten-dimensional theory of sec. 2.4.

These expressions coincide with the Kähler potentials for the moduli space of Calabi-Yau manifolds. This is a consequence of the fact that the presence of torsion does not affect the kinetic terms for the fields but only the superpotential as we already argued in section 2.4.

Using (3.21), (3.24), (3.14) and (3.16) we see that both J and Ω are not closed but obey

$$\begin{aligned} \mathrm{d}\Omega &= \left(Z^K e_{aK} + F_K m_a^K\right) \tilde{\omega}^a , \\ \mathrm{d}J &= \mathrm{Im}\, t^a \left(m_a^K \alpha_K + e_{aL} \beta^L\right) , \\ \mathrm{d}(J \wedge J) &= 0 , \end{aligned} \tag{3.29}$$

where the last equation is a direct consequence of the fact that there are no one- or five-forms in the truncated subspace. Comparing (3.29) with (2.15) we infer that the torsion classes W_1, W_2 and W_3 can be non-trivial while $W_4 = W_5 = 0$. Of course, this is precisely due to the fact that we are dropping all $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations and both W_4 and W_5 are triplets.

From (3.29) we see that the non-zero torsion is parameterized by the (constant) matrices e_{aK} and m_a^K. They can be chosen arbitrarily and only have to satisfy (3.17). Ref. [15] considered the special case $m_a^K = 0 = e_{ak}$ or in other words kept only $e_{a0} \neq 0$. From (3.29) we learn that this implies $d\Omega = Z^0 e_{a0} \tilde{\omega}^a$. Put differently, for $m_a^K = 0 = e_{ak}$, Ω satisfies

multiplet	multiplicity	bosonic field content
gravity multiplet	1	$(g_{\mu\nu}, A_1^0)$
vector multiplets	b_J	(A_1^a, t^a)
hypermultiplets	b_ρ	$(z^k, \xi^k, \tilde{\xi}_k)$
tensor multiplet	1	$(B_2^{(4)}, \phi, \xi^0, \tilde{\xi}_0)$

Table 7: $N = 2$ multiplets for Type IIA supergravity compactified on Y.

additionally $\mathrm{d}\,\mathrm{Im}\,\Omega = 0$. This in turn implies that the torsion class $W_1 \oplus W_2$ is real and such $SU(3)$ manifolds are called half-flat [45].

Let us now turn to the Ramond–Ramond sector and discuss the truncation of the ten-dimensional fields. As we will see local gauge invariance gives a separate argument for the relations (3.14) and (3.16). Since the Ramond sector differs for type IIA and type IIB we discuss both case in turn. Let us start with type IIA.

3.3 The reduction of the type IIA RR-sector

The RR-sector of the ten-dimensional type IIA supergravity contains a one-form A_1 and a three-form A_3.[16] Since we are projecting out the triplets we see from table 2 that A_1 only contains a singlet which again trivially decends to the four-dimensional theory. This four-dimensional vector field is commonly denotes by A_1^0 since it is related to the graviphoton in the gravitational multiplet.[17]

The three-form gauge potential A_3 is expanded into the basis ω_a and (α_K, β^L) introduced in (3.9) and (3.6) as

$$A_3 = A_1^a \wedge \omega_a + \xi^K \alpha_K + \tilde{\xi}_L \beta^L \ . \qquad (3.30)$$

As before the coefficents in this expansion correspond to dynamical fields in the four-dimensional effective action. The A_1^a denote b_J four-dimensional vectors (or one-forms) while $\xi^K, \tilde{\xi}_L$ are $2b_\rho + 2$ scalars. Together with the fields from the NS-sector discussed in the previous section they assemble into $N = 2$ multiplets as shown in table 7. This table is the four-dimensional 'effective' version of table 5.

The spectrum looks the same as the $N = 2$ spectrum obtained in Calabi-Yau compactifications [3, 64]. The difference here is that the expansion (3.30) is no longer in terms of harmonic forms on Y but instead in terms of the forms which obey (3.14), (3.16). As a consequence the fields are no longer massless or in other words the forms $\omega^a, \alpha_K, \beta^K, \tilde{\omega}^a$ are no longer zero modes of the Laplace operator. Instead they are eigenvectors of Δ with eigenvalues given by their masses.

[16] As we already discussed in section 2.5 there are commonly two different field basis for the p-form gauge potentials used. They are related by $C_p = e^B A_p$ and as a consequence the definition of their field strength differs. In type IIA it is more convenient to use the A-basis.

[17] Once additional vector multiplets are present the graviphoton is only defined up to symplectic rotations and thus, as we will see, A_1^0 is a component in a symplectic vector.

The field strength of the ten-dimensional gauge potentials A_1, B, A_3 are defined as

$$G_2 = \mathrm{d}A_1 \ , \qquad H = \mathrm{d}B \ , \qquad G_4 = \mathrm{d}A_3 \ , \qquad F_4 = G_4 + B \wedge G_2 \ . \tag{3.31}$$

Since we want to include background fluxes we split the field strengths into an exact piece plus a flux term. Explicitly we have

$$\begin{aligned} G_2 &= G_2^{\mathrm{fl}} + \mathrm{d}A_1 \ , \qquad H = H^{\mathrm{fl}} + \mathrm{d}B \ , \\ F_4 &= G_4^{\mathrm{fl}} - H^{\mathrm{fl}} \wedge A_1 + B \wedge G_2^{\mathrm{fl}} + \mathrm{d}A_3 + B \wedge \mathrm{d}A_1 \ . \end{aligned} \tag{3.32}$$

The background fluxes can also be expanded in the truncatd basis as

$$G_2^{\mathrm{fl}} = m^a_{\mathrm{RR}}\, \omega_a \ , \qquad G_4^{\mathrm{fl}} = e_{\mathrm{RR}\, a}\, \tilde{\omega}^a \ , \qquad H^{\mathrm{fl}} = m_0^K\, \alpha_K + e_{0\,K}\, \beta^K \ , \tag{3.33}$$

which defines the RR-flux parameter $e_{\mathrm{RR}\,a}, m^a_{\mathrm{RR}}$ and the NS-flux parameter $e_{0\,K}, m_0^K$. We have choosen this notation for the NS-fluxes since, as we will see, they naturally combine with the torsion parameters $e_{a\,K}, m_a^K$ to form the matrices

$$e_{AK} = (e_{0\,K}, e_{aK}) \ , \qquad m_A^K = (m_0^K, m_a^K) \ , \qquad A = 0, \ldots, b_J \ . \tag{3.34}$$

In analogy with (3.17) the fluxes (3.33) also have to satisfy a consistency condition. They should be choosen such that $\mathrm{d}F^{\mathrm{fl}} = \mathrm{d}^\dagger F^{\mathrm{fl}} = 0$ for all fluxes. This results in the conditions

$$m^a_{\mathrm{RR}}\, m_a^K = 0 = m^a_{\mathrm{RR}}\, e_{aK} \ , \qquad e^a_{\mathrm{RR}}\, m_a^K = 0 = e^a_{\mathrm{RR}}\, e_{aK} \ , \qquad V_0 \cdot V_a = 0 \ , \tag{3.35}$$

where the symplectic vectors V and their symplectic inner product are defined in (3.17).

Let us come back to the field strength (3.32) and discuss their four-dimensional gauge invariance. The four-dimensional theory has a standard gauge invariance associated with the $(b_J + 1)$ Abelian gauge bosons $A_1^A = (A_1^0, A_1^a)$ and a two-form gauge invariance associated with the four-dimensional NS two-form $B_2^{(4)}$

$$A_1^A \to A_1^A + \mathrm{d}\Theta^A \ , \qquad B_2^{(4)} \to B_2^{(4)} + \mathrm{d}\Theta_1 \ , \tag{3.36}$$

where Θ^A are scalar gauge parameters while Θ_1 is an independent one-form gauge parameter. From (3.32) we see that both G_2 and H are gauge invariant but, at first sight, F_4 is not. Furthermore, also $\mathrm{d}A_3$ is naively not invariant under (3.36). This can be seen by inserting the expansion (3.30) into $\mathrm{d}A_3$ which yields

$$\begin{aligned} \mathrm{d}A_3 &= \mathrm{d}A_1^a \wedge \omega_a + A_1^a \wedge \mathrm{d}\omega_a + \mathrm{d}\xi^K \wedge \alpha_K + \xi^K \mathrm{d}\alpha_K + \mathrm{d}\tilde{\xi}_L \wedge \beta^L + \tilde{\xi}_L \mathrm{d}\beta^L \ , \\ &= \mathrm{d}A_1^a \wedge \omega_a + (\mathrm{d}\xi^K + A_1^a\, m_a^K) \wedge \alpha_K + \xi^K\, \mathrm{d}\alpha_K \\ &\qquad\qquad + (\mathrm{d}\tilde{\xi}_L + A_1^a e_{aL}) \wedge \beta^L + \tilde{\xi}_L\, \mathrm{d}\beta^L \ , \end{aligned} \tag{3.37}$$

where in the second equation we inserted (3.14). The terms including A_1^a violate the gauge invariance (3.36).

In order to recover gauge invariance we have to modify the transformation laws. The two-form gauge invariance can be maintained by assigning the transformations

$$A_1^a \to A_1^a - m^a_{\mathrm{RR}} \Theta_1 \tag{3.38}$$

to the vectors. This transformation implies that one linear combination of vectors is pure gauge or in other words this linear combination can be 'eaten' by the two-form $B_2^{(4)}$. As a consequence $B_2^{(4)}$ becomes massive by a Stueckelberg-type mechanism as already observed in [64].

The local one-form gauge invariance of A_1^A can be recovered by assigning Peccei-Quinn type transformations to the RR-scalars $\xi^K, \tilde{\xi}_L$

$$\xi^K \to \xi^K - \Theta^A m_A^K \ , \qquad \tilde{\xi}_L \to \tilde{\xi}_L - \Theta^A e_{AL} \ . \tag{3.39}$$

In this case an appropriate fraction of the scalars $\xi^K, \tilde{\xi}_L$ can be eaten by the gauge bosons or in other words ξ^K and $\tilde{\xi}_L$ are the appropriate Goldstone bosons which render some of the A_1^A massive.

Note that gauge invariance can only be maintained if we impose (3.14) with m_A^K and e_{AL} being constant matrices. We could repeat the same argument in the dual formulation of type IIA [80] where instead of A_3 the dual gauge potential A_5 appears. In this case gauge invariance leads to the constraints (3.16) together with $\mathrm{d}\tilde{\omega}^a = 0$ of (3.14). Thus also from this point of view one can motivate the differential relations (3.14), (3.16) which is essentially the argument given in [72].

From (3.39) we also see that the torsion and the NS-fluxes precisely play the role of Killing vectors which gauge translational isometries of the RR scalars ξ^K and $\tilde{\xi}_L$ in the hypermultiplet sector. Via the standard relations of gauged supergravity [74] this induces a scalar potential as was worked out explicitly, for example, in refs. [64, 15]. The magnetic RR-fluxes instead render the four-dimensional antisymmetric tensor $B_2^{(4)}$ massive and also induce a scalar potential [64].[18]

Before turning to the reduction of type IIB let us summarize the situation so far. We truncated the ten-dimensional spectrum by insisting that there are only two gravitinos in the gravitational multiplet and a finite number of modes in the effective action. This corresponds to expanding the ten-dimensional fields in terms of basis defined in (3.9) and (3.6). In addition we required local gauge invariance in the effective action which independently led to the conditions (3.14) and (3.16).

The same analysis can be repeated for type IIB supergravity to which we now turn.

3.4 The reduction of the type IIB RR-sector

In IIB supergravity the RR sector contains a scalar l $(= C_0)$, a two form C_2 and a four-form C_4. Exactly as in the previous section these fields are expanded in terms of the finite basis defined in (3.9) and (3.6) as follows

$$\begin{aligned} C_2 &= C_2^{(4)} + c^a\, \omega_a \ , \\ C_4 &= D_2^a \wedge \omega_a + V^K \wedge \alpha_K - U_K \wedge \beta^K + \rho_a\, \tilde{\omega}^a \ . \end{aligned} \tag{3.40}$$

From a four-dimensional point of view $C_2^{(4)}, D_2^a$ are two-forms, V^K, U_K are one-forms and ρ_a are scalars. The field strength F_5 of C_4 is self-dual which eliminates half of the degrees

[18] Although not visible form the discussion, here let us add that the electric RR-fluxes play the role of Green-Schwarz-type couplings [64].

multiplet	multiplicity	bosonic field content
gravity multiplet	1	$(g_{\mu\nu}, V^0)$
vector multiplets	b_ρ	(V^k, z^k)
hypermultiplets	b_J	(t^a, c^a, ρ_a)
double-tensor multiplet	1	$(B_2^{(4)}, C_2^{(4)}, \phi, l)$

Table 8: $N = 2$ multiplets for Type IIB supergravity compactified on Y.

of freedom in the expansion of C_4. Conventionally one chooses to eleminate D_2^a and U_K in favour of ρ_a and V^K. Together with the fields from the NS sector all fields assemble into the $N = 2$ multiplets given in table 8 which is the four-dimensional 'effective' version of 6.[19]

The field strengths of the ten-dimensional gauge potentials C_2, B and C_4 are defined as

$$H = \mathrm{d}B \ , \qquad F_3 \ = \ \mathrm{d}C_2 - l\mathrm{d}B \ , \qquad F_5 = \mathrm{d}C_4 - H \wedge C_2 \ . \tag{3.41}$$

Separating again the fluxes from the exact piece we arrive at

$$\begin{aligned} H = H^{\rm fl} + \mathrm{d}B \ , \qquad F_3 \ &= \ G_3^{\rm fl} - lH^{\rm fl} + \mathrm{d}C_2 - l\mathrm{d}B \ , \\ F_5 = B \wedge G_3^{\rm fl} - H^{\rm fl} &\wedge C_2 + \mathrm{d}C_4 - H \wedge C_2 \ . \end{aligned} \tag{3.42}$$

As before the fluxes are expanded in terms of the truncated basis as

$$G_3^{\rm fl} = \tilde{m}_{\rm RR}^K \, \alpha_K + \tilde{e}_{{\rm RR}\,K} \, \beta^K \ , \qquad H^{\rm fl} = \tilde{m}_0^K \, \alpha_K + \tilde{e}_{0K} \, \beta^K \ , \tag{3.43}$$

where we use $\tilde{e}, \tilde{m}$ to denote the fluxes in type IIB. In this case consistency requires $\mathrm{d}F_3^{\rm fl} = \mathrm{d}H^{\rm fl} = 0$ which translates into

$$V_0 \cdot V_a = 0 \ , \qquad V_{\rm RR} \cdot V_a = 0 \ , \tag{3.44}$$

where we defined the symplectic vector $V_{\rm RR} = (\tilde{e}_{{\rm RR}\,K}, \tilde{m}_{\rm RR}^K)$.

In type IIB the four-dimensional theory has a standard one-form gauge invariance associated with the $(b_\rho + 1)$ Abelian gauge bosons V^K and their magnetic duals U_K. In addition there are $(b_J + 2)$ two-form gauge transformations.[20] Together they read

$$\begin{aligned} &V^K \to V^K + \mathrm{d}\Theta_V^K \ , \qquad U_K \to U_K + \mathrm{d}\Theta_K^U \ , \\ &B_2^{(4)} \to B_2^{(4)} + d\Theta_1^{(B)} \ , \qquad C_2^{(4)} \to C_2^{(4)} + d\Theta_1^0 \ , \qquad D_2^a \to D_2^a + d\Theta_1^a \ . \end{aligned} \tag{3.45}$$

[19] If we choose to eliminate the scalars ρ_a in favor of the four-dimensional tensors D_2^a the b_J hypermultiplets are replaced by b_J tensor multiplets (t^a, c^a, D_2^a).

[20] The issue of gauge invariance is best discused in terms of the tensor multiplets (t^a, c^a, D_2^a) where the ρ_a are eliminated in favor of the four-dimensional tensors D_2^a and thus the b_J hypermultiplets are traded for b_J tensor multiplets.

	IIA	IIB
electric RR-flux $e_{\rm RR}$	Green–Schwarz coupling	Green-Schwarz coupling
magnetic RR-flux $m_{\rm RR}$	massive tensor $B_2^{(4)}$	massive tensor $B_2^{(4)}$
electric NS-flux e_0	massive A_1^0	massive A_1^0
magnetic NS-flux m_0	massive A_1^0	massive tensor $C_2^{(4)}$
electric torsion e_{aK}	massive A_1^a	massive A_1^K
magnetic torsion m_a^K	massive A_1^a	massive tensors D_2^a

Table 9: Effect of fluxes and torsion.

From (3.42) we see that H and F_3 are invariant but F_5 is not. Exactly as in type IIA also $\mathrm{d}C_4$ is not gauge invariant. Inserting (3.40) into $\mathrm{d}C_4$ we arrive at

$$\begin{aligned} \mathrm{d}C_4 &= \mathrm{d}D_2^a \wedge \omega_a + D_2^a \wedge \mathrm{d}\omega_a + \mathrm{d}V^K \wedge \alpha_K - V^K \wedge \mathrm{d}\alpha_K \\ &\qquad - \mathrm{d}U_K \wedge \beta^K + U_K \wedge \mathrm{d}\beta^K + \mathrm{d}\rho_a \wedge \tilde{\omega}^a + \rho_a \mathrm{d}\tilde{\omega}^a \\ &= \mathrm{d}D_2^a \wedge \omega_a + (D_2^a \tilde{m}_a^K + \mathrm{d}V^K) \wedge \alpha_K + (D_2^a \tilde{e}_{aK} + \mathrm{d}U_K) \wedge \beta^K \\ &\qquad + (\mathrm{d}\rho_a - \tilde{e}_{aK} V^K - \tilde{m}_a^K U_K) \wedge \tilde{\omega}^a \ , \end{aligned} \tag{3.46}$$

where in the second equation we inserted (3.14), (3.16). The terms in the last line involving V^K, U_K explicitly violate the gauge invarianve (3.45).

As in IIA, local gauge invariance necessitates (3.14) and (3.16) in which case the transformation laws (3.45) can be modified according to

$$\begin{aligned} &V^K \to V^K + \tilde{m}_A^K\,\Theta_1^A \ , \qquad U_K \to U_K + \tilde{e}_{AK}\Theta_1^A \ , \\ &V^K \to V^K + \tilde{m}_{\rm RR}^K \Theta_1^{(B)} \ , \qquad U_K \to U_K + \tilde{e}_{{\rm RR}\,K}\Theta_1^{(B)} \ , \\ &\rho_a \to \rho_a + \tilde{e}_{aK}\Theta_V^K + \tilde{m}_a^L \Theta_L^U \ . \end{aligned} \tag{3.47}$$

With these transformations gauge invariance is recovered. We see that for a non-vanishing magnetic matrix $\tilde{m}_A^K$ or a non-vanishing magnetic RR-flux $\tilde{m}_{\rm RR}^K$ some of the electric gauge bosons V^K become pure gauge degrees of freedom which can be absorbed (eaten) into $C_2^{(4)}, B_2^{(4)}$ or D_2^a. Put differently, these antisymmetric tensors gain longitudinal degrees of freedom by a Stueckelberg mechanism or in other words they become massive. Note that on the type IIA side this only happens for magnetic RR-fluxes but not for NS-fluxes or torsion. Instead in type IIB it occurs for all magnetic fluxes and torsion. This situation is summarized in table 9.

For electric fluxes some of the scalars ρ_a become Goldstone bosons and are eaten by the corresponding vectors while for electric RR-fluxes a Green-Schwarz type coupling is induced [64]. We will also see these facts in the supersymmetric transformation law of the gravitinos to which we now turn.

3.5 Relation to gauged supergravity

In section 2.5 we computed the transformation of the gravitinos which reside in the gravitational multiplet. From this we read off the matrix S_{AB} and via (2.114) the three Killing prepotentials $\mathcal{P}^x$ which can be viewed as the $N=2$ version of the superpotential and the D-term. The purpose of this section is to truncate these results to the finite basis (3.9) and (3.6) and then compare with the formulas of gauged $N=2$ supergravity.

In section 2.5 we gave $S^{(4)}_{AB}$ in two different forms. In (2.126) and (2.132) we used a ten-dimensional field basis while in (2.136) and (2.137) we essentially already used a four-dimensional field basis. To be more precise we performed a Weyl rescaling in the ten-dimensional action and introduced the four-dimensional dilaton $\phi^{(4)}$ but kept the dependence on all ten space-time coordinates. Now we merely need to truncate (2.136) and (2.137) (or equivalently (2.139) and (2.140)) to the finite-dimensional subspace of light modes and integrate over the compact manifold Y.

Using the truncation defined in 3.1–3.4 we infer from (2.139) for type IIA

$$
\begin{aligned}
(\mathcal{P}^1 + i\mathcal{P}^2) &= -2\,\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}} \int_Y \langle \Phi^+, \mathrm{d}\Phi^- \rangle \ , \\
\mathcal{P}^3 &= \frac{1}{\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \int_Y \langle \Phi^+, G_{\mathrm{A}} \rangle \ ,
\end{aligned}
\tag{3.48}
$$

where $\Phi^+ = c\,\mathrm{e}^{-t}$, $\Phi^- = \Omega$. Inserting the separation (3.32) of G_A into flux terms and field strength we arrive after partial integration at

$$
\begin{aligned}
(\mathcal{P}^1 + i\mathcal{P}^2) &= -2c\,\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}} \int_Y \left(\mathrm{d}t + H^{\mathrm{fl}}\right) \wedge \Omega \ , \\
\mathcal{P}^3 &= \frac{c}{\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \int_Y e^{-t} \wedge (G^{\mathrm{fl}}_{2n} + \mathrm{d}A_{2n-1} - H^{\mathrm{fl}} \wedge A_{2n-3}) \ ,
\end{aligned}
\tag{3.49}
$$

where $t = B + \mathrm{i}J$ and $e^t \wedge G^{\mathrm{fl}}_{2n} = \frac{1}{6}G^{\mathrm{fl}}_0 t^3 + \frac{1}{2}G^{\mathrm{fl}}_2 t^2 + G^{\mathrm{fl}}_4 t + G^{\mathrm{fl}}_6$. We can go one step further by inserting the expansions (3.29) and (3.33) into (3.49) and performing the integrals. This yields

$$
\begin{aligned}
(\mathcal{P}^1 + \mathrm{i}\mathcal{P}^2) &= -2\,\mathrm{e}^{\frac{1}{2}K_\rho+\phi^{(4)}}\, X^A(e_{AK}Z^K + m_A^K F_K) \ , \\
\mathcal{P}^3 &= -\frac{c}{\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \left[X^A(e_{AK}\xi^K + m_A^K \tilde{\xi}_K) + (X^A e_{\mathrm{RR}\,A} + \mathcal{F}_A m^A_{\mathrm{RR}})\right] \ ,
\end{aligned}
\tag{3.50}
$$

where we used (3.34) and $X^A = (c, -ct^a)$. (3.50) can be compared with the generic structure of $\mathcal{P}^x$ as dictated by $N=2$ supergravity. In this case one has the generic expression [74, 75]

$$
\mathcal{P}^x = X^A P^x_A + \omega^x_\alpha (e^\alpha_A X^A - m^{\alpha A}\mathcal{F}_A) \ ,
\tag{3.51}
$$

where P^x_A are the Killing prepotentials which only depend on scalars in $N=2$ hypermultiplets. The second term in (3.51) arises when tensor multiplets are present and e^α_A denotes possible Green-Schwarz-type couplings while $m^{\alpha A}$ are related to mass terms of antisymmetric two-tensors. (ω^x_α is an appropriate $SU(2)$ connection on the hypermultiplet moduli space, see [75] for further details.) Comparing (3.50) and (3.51) we see that they are consist

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with each other and furthermore for $m_{\rm RR}^A \neq 0$ we necessarily have massive tensor fields as was already observed in section 3.3 and refs. [64, 75].

We can repeat the same analysis for type IIB supergravity starting from (2.140) and obtain

$$\begin{aligned}
(\mathcal{P}^1 - \mathrm{i}\mathcal{P}^2) &= -2\,\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}} \int_Y \langle \Phi^-, \mathrm{d}\Phi^+ \rangle = -2c\,\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}} \int_Y \Big(\mathrm{d}t + H^{\rm fl}\Big)\wedge\Omega\ , \\
&= -2\,\mathrm{e}^{\frac{1}{2}K_J+\phi^{(4)}} \left(Z^K \tilde e_{AK} X^A + F_K \tilde m_A^K X^A\right)\ , \\
\mathcal{P}^3 &= \frac{1}{\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \int_Y \langle \Phi^-, G_{\rm B} \rangle = \frac{1}{\sqrt{2}}\,\mathrm{e}^{2\phi^{(4)}} \int_Y \Big(G_3^{\rm fl} - lH^{\rm fl} + \mathrm{d}C_2 - l\mathrm{d}B\Big)\wedge\Omega \\
&= \frac{1}{\sqrt{2}}\,e^{2\phi^{(4)}} \left(Z^K(\tilde e_{{\rm RR}\,K} - \tilde e_{AK}\xi^A) + F_K(\tilde m_{\rm RR}^K - \tilde m_A^K \xi^A)\right)\ ,
\end{aligned} \tag{3.52}$$

where we defined

$$\xi^A = (\xi^0, \xi^a) = (l, c^a - lb^a)\ , \qquad \tilde e_{AK} = (\tilde e_{0K}, \tilde e_{aK})\ , \quad \tilde m_A^K = (\tilde m_0^K, \tilde m_a^K)\ . \tag{3.53}$$

The comparison with $N = 2$ supergravity essentially uses again (3.51) but since in type IIB the scalars t^a and z^k in vector and hypermultiplets are interchanged compared to type IIA (3.51) has be replaced by

$$\mathcal{P}^x = Z^K P_K^x + \omega_\alpha^x (e_K^\alpha Z^K - m^{\alpha K} F_K)\ . \tag{3.54}$$

Comparing (3.52) with (3.54) we see that for electric fluxes $\tilde e_{{\rm RR}\,K}, \tilde e_{AK}$ we have a standard gauged supergravity while for magnetic fluxes $\tilde m_{\rm RR}^K, \tilde m_A^K$ we have in addition massive antisymmetric tensors [64, 75, 72]. This was already observed at the end of section 3.4 and is summarized in table 9.

3.6 Mirror symmetry

Now that we have discussed the supersymmetry transformation in terms of fluxes and torsion let us return to the issue of mirror symmetry. For Calabi-Yau manifolds the mirror conjectures states that for a given Calabi-Yau Y there exists a mirror manifold $\tilde Y$, with the property that the Hodge numbers and Kähler and complex structure deformations are exchanged in passing from Y to $\tilde Y$. In particular the moduli spaces $\mathcal{M}_J$ for one manifold and $\mathcal{M}_\rho$ for the other (as well as their respective prepotentials) are identified. In string theory mirror symmetry manifests itself in the equivalence of type IIA compactified on Y and type IIB compactified on the mirror manifold $\tilde Y$. In particular this states that the respective low energy effective Lagrangians are identical for compactifications on mirror manifolds. In the notation used in the previous section this amounts to the identification [4]

$$X^A \leftrightarrow Z^K\ , \qquad \mathcal{F}^A \leftrightarrow F^K\ , \qquad \xi^A \leftrightarrow \xi^K\ , \tag{3.55}$$

on mirror pairs.

Of course it is an interesting question to see what happens to this symmetry for manifolds with $SU(3)$ structure and in particular to what extent the exchange (3.55) also holds on a generalized pair of mirror manifolds. It was proposed in [23] that mirror symmetry for manifolds of $SU(3)$ structure amounts to the exchange of the two pure spinors together with an exchange of even and odd RR-forms

$$e^{-B-\mathrm{i}J} \leftrightarrow \Omega , \qquad G_{\text{even}} \leftrightarrow G_{\text{odd}} . \tag{3.56}$$

As we already discussed at the end of section 2.5, (3.56) is a special case of the more general map (2.141) which we expect to hold if instead of a $SU(3)$ structure one repeats the computation for a $\mathrm{SU}(3)\times\mathrm{SU}(3)$-structure. Using the expansions (3.20), (3.24), (3.30), (3.40) and (3.53) one readily verifies that the map (3.56) implies (3.55) if no fluxes are turned on.

In Calabi-Yau compactifications with only electric and magnetic RR-fluxes arising in (3.33) from the IIA RR-field strength G_{2n}^{fl} and in (3.43) from IIB G_3^{fl}, it was indeed observed in refs. [64] that at the level of the effective action mirror symmetry is straightforwardly realized as the exchange (3.55) together with an exchange of the flux parameters

$$e_{RR} \leftrightarrow \tilde{e}_{RR} , \qquad m_{RR} \leftrightarrow \tilde{m}_{RR} . \tag{3.57}$$

Furthermore for the case of RR fluxes on manifolds with $SU(3)$ structure, it was shown in [28] that the supersymmetry equations are symmetric under the exchange (3.56). This is also reflected in the Killing prepotentials derived in this paper. From Eqs. (2.139) and (2.140) we see that the two (ten-dimensional) $\mathcal{P}^3$ are symmetric under (3.56). Similarly, Eqs. (3.49) and (3.52) show that in the truncated theory the two $\mathcal{P}^3$s are exchanged under the map (3.55) together with (3.57), i.e. exactly in the same way as in Calabi-Yau compactifications with RR fluxes.

The situation for NS-fluxes is more involved. On the one hand, the expressions for the Killing prepotentials $\mathcal{P}^1$ and $\mathcal{P}^2$ in Eqs (2.139) and (2.140), look perfectly mirror symmetric, i.e. respect the exchange (2.141). Note however that this exchange does not map H-flux (which appears in $\mathrm{d}\Phi^+$) to itself, but rather to torsion components in $\mathrm{d}\Phi^-$. Nevertheless, as we mentioned at the end of section 2.5, these expressions were obtained for the particular case of a single $SU(3)$ structure, where Φ^- contains only a 3-form and not all odd forms. In this case, from $\mathrm{d}\Phi^+$, the 3-form $\mathrm{d}(B+iJ)$ contributes to the superpotential, while the five-form $\mathrm{d}(B+iJ)^2$ does not, since there is no 1-form in Φ^-.

After performing the truncation, we can see explicitly by comparing (3.50) and (3.52) that H^{fl} is not mapped to itself. Indeed, the type IIA NS-fluxes e_{0K}, m_K^0 arising from H^{fl} in (3.33) and the type IIB NS-fluxes $\tilde{e}_{0K}, \tilde{m}_K^0$ arising from H^{fl} in (3.43) are not interchanged under mirror symmetry. Instead for electric fluxes they are mapped to the torsion coefficients $e_{A0}, \tilde{e}_{A0}$ of half-flat manifolds [15, 18, 36]. In fact this immediately generalizes for the entire electric matrices $e_{AK}, \tilde{e}_{KA}$ defined in (3.34) and (3.53). We find perfect agreement under mirror symmetry if we perform the map (3.55) and simultaneously exchange the electric matrices

$$e_{AK} \leftrightarrow \tilde{e}_{KA} . \tag{3.58}$$

For the magnetic matrices $m_A^K, \tilde{m}_A^K$ no mirror symmetry is observed in eqs. (3.50) and (3.52). In this case antisymmetric tensor fields become massive on the IIB side while on the IIA side this is not the case. As we already stated before we expect that such a contribution naturally arises when the backgrounds with $SU(3)$ structure are replaced by the more general backgrounds with $SU(3) \times SU(3)$ structure. Work along these lines is in progress.

4. Conclusions

In this paper we have shown that type II supergravities in space-time backgrounds which admit an $SU(3) \times SU(3)$ structure share many features with four-dimensional $N = 2$ gauged supergravities. The reason for this is that in such backgrounds the ten-dimensional Lorentz symmetry $SO(1,9)$ can be replaced by a symmetry $SO(1,3) \times SU(3) \times SU(3)$ and one can consistently write the theory in terms of only eight out of the 32 original supercharges. Following the approach pioneered in [69], this in turn allowed us to rewrite the ten-dimensional theory in a form which strongly resembles the four-dimensional $N = 2$ gauged supergravities but without the need of any Kaluza–Klein truncation. For simplicity, in many parts of the paper we concentrated on the special situation where there is a single SU(3) structure (or in other words where the two SU(3) structures coincide). However, given the covariant form of the final expressions we expect that they hold for general SU(3) × SU(3) structures.

In particular we showed, using results in [48], that the metric deformations together with B can be viewed as coordinates of a product of special Kähler manifolds. The corresponding Kähler potentials can be expressed in terms of a pure spinor $\Phi^{\pm}$ of Spin(6, 6) and its complex conjugate and they coincide with the corresponding Hitchin functionals. In the SU(3) case the two pure spinors are $\Phi^+ = e^{-B-\mathrm{i}J}$ and $\Phi^- = \Omega$ and are constructed as bispinors of the $SU(3)$-singlet spinors $\eta_{\pm}$.

We also computed the supersymmetry transformation of the gravitinos and determined the three Killing prepotential $\mathcal{P}^x$ in the case of type IIA and IIB. They are expressed (see eqns. (2.139) and (2.140)) in terms of Spin(6, 6) invariant inner products of the pure spinors $\Phi^{\pm}$ with the RR fluxes and with $\mathrm{d}\Phi^{\mp}$ which encode the intrinsic torsion and H-flux.

By further breaking the $SU(2)_R$ symmetry of the $N = 2$ down to a $U(1)_R$, we were able to find the D-term and superpotential of an $N = 1$ theory. They depend on two angles that parameterize the $N = 2 \to N = 1$ reduction. In this way we obtained the most general $N = 1$ superpotential for manifolds admitting an $SU(3)$ structure which contains in particular all the cases that have been studied so far. The case of the heterotic string compactified on manifolds with SU(3) structure will be presented in [88].

An important caveat to the reformulation was that we explicitly dropped the multiplets containing additional spin-$\frac{3}{2}$ fields, present because that underlying theory really has $N = 8$ supersymmetry. In particular, including these multiplets should modify the scalar field moduli space. This is related to the fact that there is actually no natural special Kähler geometry on the physical scalar degrees of freedom, but only on the larger space of SU(3)× SU(3) structures. In the full formulation, it must be possible to gauge away some of the degrees of freedom in $\Phi^{\pm}$. Equally, there should actually be an underlying $N = 8$ type theory, where all the supersymmetries are kept and the scalar fields parameterize a $E_{7(7)}/\,\mathrm{SU}(8)$ coset.

In the second part of the paper we demanded that the ten-dimensional manifold had a product structure and performed a truncation of the spectrum. This reduced theory was then shown to be consistent with a four-dimensional $N = 2$ gauged supergravity. In particular the gauged isometries are translational isometries of the hypermultiplet and tensor multiplet moduli space with gauge charges or mass parameters given by the fluxes and the torsion. Electric fluxes (RR, NS or torsion) give masses to the vectors coming from the three- or four-form RR potential for IIA or IIB, resulting in a standard $N = 2$ gauged supergravity, while magnetic fluxes give mass to the antisymmetric tensors by a Stueckelberg-type mechanism.

The truncation of the spectrum done in section 3 also excluded spin-$\frac{3}{2}$ multiplets which amounted to projecting out $\mathbf{3} + \bar{\mathbf{3}}$ representations. This sets the torsion classes W_4 and W_5 to zero. Additionally, no warp-factor was allowed. It turns out that allowing for these torsion classes or a warp factor is not straightforward. A first step in that direction was taken in refs. [89, 90], which considered KK-reductions in warped products with a conformal Calabi-Yau factor. In particular, it was claimed in [89] that the warp factor affects the $N = 1$ Kähler potential, but not the superpotential. It would be interesting to generalize our results allowing for such warped compactifications.

As we mentioned, for simplicity we mostly considered the case of a single SU(3) structure instead of the more general situation of a SU(3) × SU(3) structures. It should not be too difficult to generalize our results to the generic situation. The main difference is that in this case there are globally defined vectors (given by bilinears involving the two spinors with one gamma matrix) which are nowhere vanishing if there is a common SU(2) substructure. We expect that the Kähler potential (2.73) and the Killing prepotentials (2.139) and (2.140) take the same form in this generalized set-up but are now evaluated with $\Phi^{\pm}$ defining SU(3)×SU(3) structures. These formulas further suggest that in the framework of SU(3)×SU(3) structures mirror symmetry with fluxes is restored and the missing magnetic fluxes can be located. We will return to these questions elsewhere.

One important question we have not addressed directly is the connection between the formulation we have discussed here and topological string theories. The target space theories of the topological A and B models are theories of deformations of complexified Kähler structure $B + \mathrm{i}J$ or complex structure ρ called Kähler and Kodaira–Spencer gravity respectively. It has recently been argued [53 – 55] that these are equivalent to theories based on the Hitchin functionals $H(\chi^+)$ and $H(\chi^-)$ where $\chi^+ = \mathrm{Re}(c\,\mathrm{e}^{-B-\mathrm{i}J})$ and $\chi^- = n\rho$. In ref. [56] the equivalence was shown to hold at one-loop for the B model, provided one considered the full functional (2.71) for odd forms rather than that based on (2.87) restricted to three-forms. In these cases the Hitchin functional is taken as the action of the theory. Crucially one must also assume that ρ and $B + \mathrm{i}J$ are closed and one only takes variations by exact forms. Hitchin's result [51] is that the equations of motion then imply that the relevant structure is integrable. Here, considering the effective theory of the physical string, we have seen that the Hitchin functionals naturally appear as Kähler potentials without the restrictions that the forms are closed. $\mathcal{N} = 2$ supersymmetric vacua require integrability of the two structures [34], while $\mathcal{N} = 1$ vacua impose integrability of only one of the two [44]. This led [44] to conjecture that there is a topological model associated to any $\mathcal{N} = 1$ vacuum. It would be interesting to understand the connection between the topological and the physical theories in more detail.

Let us end by noting that the approach presented in this paper of rewriting the ten-dimensional supergravities in a $\mathrm{SU}(3) \times \mathrm{SU}(3)$ background can equally well be applied to other theories and structure groups. Of particular interest is eleven-dimensional supergravity rewritten in a $\mathrm{SO}(1,3) \times G_2$ background. In this case the ten-dimensional theory should resemble a four-dimensional $N = 1$ theory with a Kähler potential, a superpotential and D-terms. There is also a Hitchin functional for the three-form describing the G_2 and it is natural to conjecture that this will give the exponential of the Kähler potential.

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